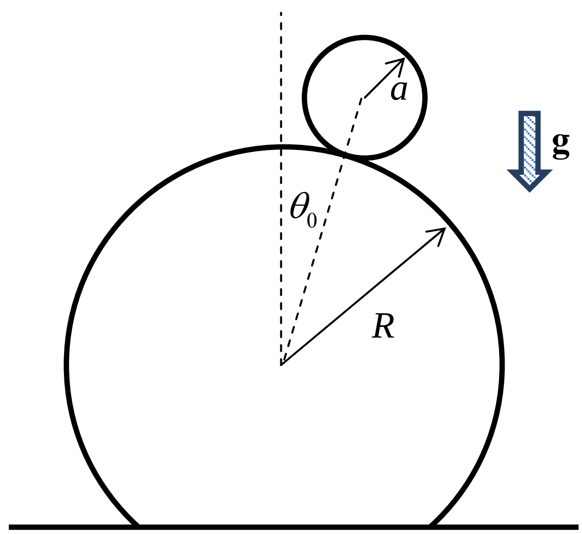
Classical Mechanics Qualifier (August 2015)
George Mason University

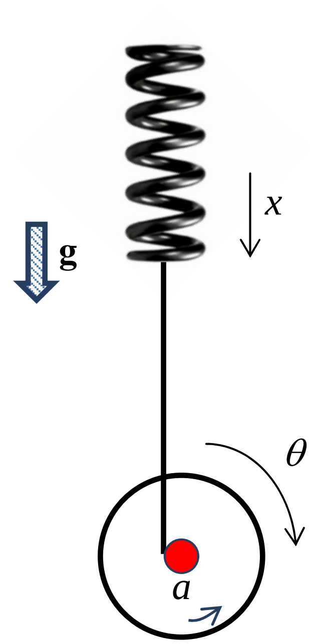
You will have **THREE** hours to complete the exam.
You are allowed to use your graduate textbook during the exam.
Choose **4 out of the 5** problems below.

Problem 1 (25pts)

A thin disk of mass *m* and radius *a* initially at rest at an angle θ_0 relative to the vertical *z*-axis through the center of a fixed cylinder of radius *R* as shown in the figure to the right. Gravity is directed downward as shown. The thin disk is released and it rolls without slipping down the side of the cylinder. It leaves the surface of the sphere at angle θ_1. Find θ_1 in terms of θ_0.



Problem 2 (25pts)



A yo-yo of mass m and moment of inertia I is attached to a string and the top of the string spring with spring constant k as shown in the figure to the right. The radius of the axel of the yo-yo is a and both the string and the spring can assumed to have negligible mass. The yo-yo rolls down the string due to gravity and the spring will oscillate. Let x measures the displacement of the spring with respect to its unstretched length and θ gives the angle of rotation with respect to the center of mass of the yo-yo.

a) Write the Lagrangian and its equations of motion for this system in terms of the generalized coordinates x and θ.
b) Find the oscillation frequency of the spring while the yo-yo rolls down.
c) In the limit of a thin axle yo-yo $\left(ma^2 \ll I\right)$, solve the equation of motion for $x(t)$. Describe the motion in this limit.

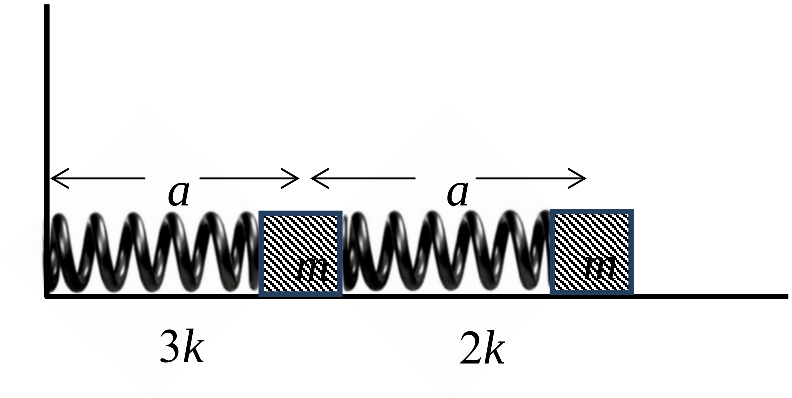
Problem 3 (25pts)

In a galaxy far far away, the Galactic Empire constructed a large spherical space station (Death Star) with mass M. Assume that the mass of Death Star is uniformly distributed and it rotates with a constant angular velocity ω_0. An imperial cruiser with mass αM landed on the surface of the space station at a location given by the colatitude θ (i.e., the latitude is $90^\circ - \theta$). Assume that the size of the cruiser to be sufficiently small such that it can simply be treated as a point mass stuck onto the surface of much larger Death Star. Also, assume α to be small.

a) What is the new moment of inertia tensor for the combined object with respect to its new center of mass and along its principle axes? [The moment of inertia for a solid sphere with respect to its center is $I_{sphere} = \frac{2}{5}MR^2$.]

b) Since the combined object is no longer spherical symmetric, the new rotational axis will now rotate in space. What is the period of precession of the rotational axis in terms of the original period of rotation $2\pi / \omega_0$?

Problem 4 (25pts)



Two equal mass *m* are attached to each other by a spring and the left mass is also attached to a fixed wall by a spring. The whole system lay flat on a frictionless surface and the spring-mass system is constrained to move only along in one direction. The unstretched equilibrium length of the springs is a. The left spring has a spring constant of $3k$ and the right spring has a spring constant of $2k$. The situation is illustrated in the diagram above.

a) Find the Lagrangian for the system.
b) Assuming small oscillations, find the resonant frequencies and normal modes for the system.
c) At t=0, the left mass is displaced from its equilibrium position by value of $-b$ to the left and the right mass is displaced from its equilibrium position by a value of $+b$ to the right. The two masses are then released from this position with zero initial velocity. Find the positions of the two masses as a function of time t.

Problem 5 (25pts)

Use the Poisson bracket to determine the value of α and β such that the following transformation is canonical,

$$Q = q^{\alpha} \cos(\beta p), \quad P = q^{\alpha} \sin(\beta p)$$

where q is a generalized coordinate and p is its associated conjugate momentum. Give a generating function for this transformation.