Classical Mechanics Qualifier Exam (August 2026)

NAME: G-NUMBER:

Important instructions: In your solutions explain the details of your derivations and solutions. Present your solutions in a clean and clear way. Show all work relevant to the solution of the problem.

(1.) Consider a particle that describes a circular orbit under the influence of an attractive central force directed toward a point on the circle.

(a) Derive the Lagrangian L in polar coordinates with a radial distance r and azimuthal angle of θ and sketch the problem.

(b) Derive the Lagrangian equations of motion.

(c) Write down the canonical momentum for θ, the equation of motion in θ-direction, and the first integral involving the constant magnitude of the angular momentum, l.

(d) Derive a second order differential equation involving r (and the constant magnitude of the angular momentum) only.

(e) Derive the equation of the orbit

$$f(r) = \frac{l}{r^2}\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{l}{mr^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right) - \frac{l}{mr}\right] \tag{1}$$

(45 points)

(2.) A Hamiltonian of one degree of freedom has the form

$$H = \frac{p^2}{2a} - bqp\ \exp(-\alpha t) + \frac{ba}{2}q^2\exp(-\alpha t)[\alpha + b\exp(-\alpha t)] + \frac{kq^2}{2}, \tag{2}$$

where a, b, α, k are constants.
Find a Lagrangian corresponding to this Hamiltonian in terms of q and $\dot{q}$, eliminating p.

(20 points)

(3.) Given the following equation:

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\boldsymbol{\nabla}\cdot\mathbf{u} = 0 \tag{3}$$

(a) What equation is this and what is the physical meaning of it? Explain each variable.

(b) Expand this equation in terms of the Cartesian coordinates (x, y, z), that is, expand the total derivate operator and the divergence terms explicitly. (Hint: $\mathbf{u} = (u, v, w)$).

(c) Using the method of linearization, prove that Eq. (3) can be written as

$$\frac{\partial\rho'}{\partial\bar{\rho}} - \frac{\mathbf{u}'\cdot\hat{\mathbf{k}}}{H} + \boldsymbol{\nabla}\cdot\mathbf{u}' = 0, \tag{4}$$

where $H = -\frac{\partial \ln\bar{\rho}}{\partial z}$ is the scale height, and $\hat{\mathbf{k}}$ is the Cartesian unit vector in the vertical direction.

(35 points)

(100 points in total.)