Classical Mechanics Qualifier (Fall 2025)
George Mason University

You are allowed to use your graduate textbook during the exam.
Four problems | Total 100 points

Problem 1 (20pts)

Consider the following Hamiltonian for a particle moving in a central force potential $U(r)$ in spherical coordinates (r,θ,ϕ),

$$H=\frac{1}{2m}\left(p_r^2+\frac{p_\theta^2}{r^2}+\frac{p_\phi^2}{r^2\sin^2\theta}\right)+U(r)$$

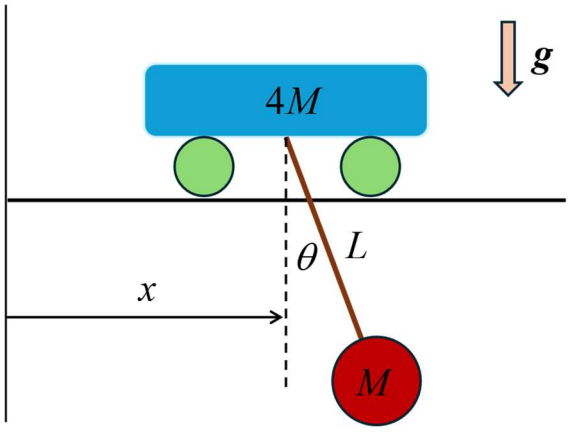
a) Show that the angular momentum about the *z*-axis $L_z=p_\phi$ is a constant of motion by explicitly calculating the Poisson bracket $\{p_\phi,H\}$ and showing that it is zero.

b) Show that the total angular momentum $L^2=p_\theta^2+\frac{p_\phi^2}{\sin^2\theta}$ is a constant of motion by explicitly calculating the Poisson bracket $\{L^2,H\}$ and showing that it is zero.

c) Use the Jacobi's identity to show that $\{L^2,L_z\}$ is also a constant of motion.

Problem 2 (40pts)

A simple pendulum with mass *M* and length *L* is suspended under a cart with mass 4*M* as shown to the right. The cart can move freely without friction on a horizontal track in the *x*-direction. Gravity is directed downward as shown.



a) Write down the Lagrangian for this system in terms of its generalized coordinates (r,θ) as shown in the diagram.

b) For small oscillations of the pendulum about $\theta=0$, find the normal frequencies of the system.

c) Find the corresponding normal modes for these small oscillations.

d) Describe qualitatively the relative motion of the cart and the pendulum for each of the normal modes found in c).

Problem 3 (40pts)

A thin disk of mass *m* and radius *a* initially at rest at an angle θ_0 relative to the vertical *z*-axis through the center of a fixed cylinder of radius *R* as shown in the figure to the right. Gravity is directed downward as shown. The thin disk is released and it rolls without slipping down the side of the cylinder. It leaves the surface of the sphere at angle θ_1. Find θ_1 in terms of θ_0.