Classical Mechanics Qualifier Exam (August, 2024)

NAME: G-NUMBER:

Important instructions:

- Clearly organize and outline your solution path and solutions.
- In your solutions explain the details of your derivations and show all work relevant to the solution path.

(1.) For internal waves in the atmosphere the nonlinear equations of motion are given in terms of the velocity $\mathbf{u} = (u, w)$, where u is the zonal wind speed (along the x-direction) and w is the vertical wind speed (along the z-direction); pressure P, internal energy equation for a gas, and the continuity equation for a Boussinesq, hence incompressible, gas. These equations are for nonrotating two-dimensional motion:

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}t} = -\frac{1}{\rho}\boldsymbol{\nabla} P + \mathbf{g} \tag{1}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0 \tag{2}$$

$$\boldsymbol{\nabla} \cdot \mathbf{u} = 0 \tag{3}$$

(a) Expand (1)–(3) in components and write in Cartesian components.

(b) In the case of small-amplitude internal waves in a fluid with no background mean flow, the total derivative (or material derivative) in (1) and (2) can be replaced with the local time rate of change. Explain the physical rationale for this simplification.

(c) Linearize (1)–(3) using the ansatz

$$\psi(x, z, t) = \bar{\psi}(z) + \psi'(x, z, t),$$

where $\psi(x, z, t)$ represents the field variables u, w, ρ, P; $\bar{\psi}$ is the background mean, and ψ' is the fluctuation component of the respective field variable. Assume stationary background mean flow and higher order fluctuation terms are negligible. State for each field variable what the linearization ansatz would be under these assumptions. Then linearize (1)–(3) componentwise (x, z). What are the linearized form of the governing equations?

(d) State the linearized form of the governing equations in matrix form.

(40 points)

(2.) Consider a particle that describes a circular orbit under the influence of an attractive central force directed toward a point on the circle.

(a) Derive the Lagrangian L in polar coordinates with a radial distance r and azimuthal angle of θ and sketch the problem.

(b) Derive the Lagrangian equations of motion.

(c) Write down the canonical momentum for θ, the equation of motion in θ-direction, and the first integral involving the constant magnitude of the angular momentum, l.

(30 points)

(3.) Given the function

$$S = \frac{m\omega}{2}(q^2 + \alpha^2)\cot(\omega t) - m\omega q\alpha \csc(\omega t) \tag{4}$$

Show that S is a solution of the Hamilton-Jacobi for Hamilton's principal function for the linear harmonic oscillator with

$$H = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \tag{5}$$

(30 points)

(100 points in total.)