Classical Mechanics Qualifier (Fall 2022)
George Mason University

You are allowed to use your graduate textbook during the exam.
Four problems | Total 100 points

Problem 1 (20pts)

A particle with mass *m* is moving in the *x*-y plane described by three different Lagrangians,

$$L_1 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}ky^2$$
$$L_2 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}k\left(x^2 + y^2\right)$$
$$L_3 = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2\right) - \frac{1}{2}k(t)\,y^2$$

where the $k(t)$ in L_3 is a function of time.

a) For each of the scenarios, determine which, if any, of the following quantities are conserved: momentum in the x direction p_x, momentum in the y direction p_y, angular momentum along the z direction L_z, and the total energy E.
b) For each conserved quantity, state the transformation under which the system is invariant.

Problem 2 (20pts)

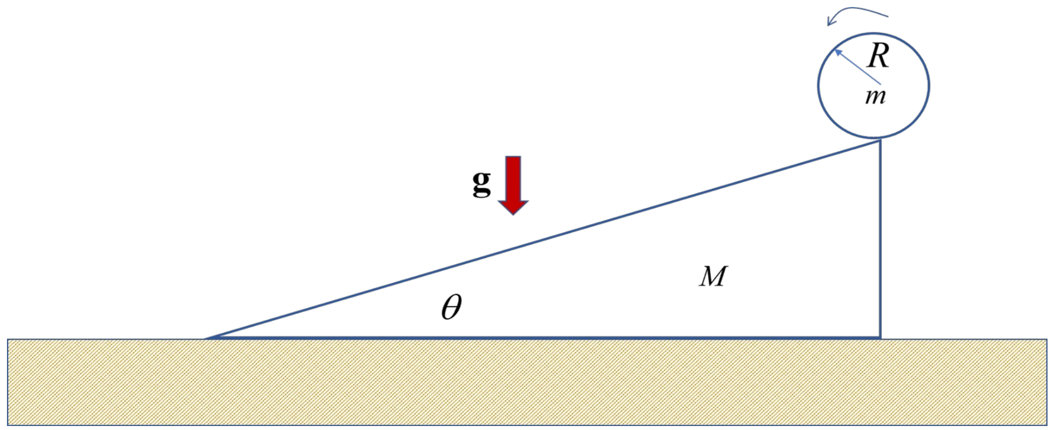
A point particle moving around a black hole can be described by the following central force potential modified from the standard Keplerian case,

$$V_{BH}(r) = -\frac{1}{r} - \frac{l^2}{r^3}$$

where l is the angular momentum of the system and for simplicity, we have normalized the system so that k=1 for the Keplerian term ($-k/r$) in the potential and μ=1 for the reduced mass.

a) Show that there are no circular orbits if $l^2<12$ and there are two if $l^2>12$.
b) Sketch a plot for the effective potential V_{eff} of the problem for the above two cases $l^2<12$ and $l^2>12$.
c) Describe qualitatively the set of possible orbits for the two different cases $l^2<12$ and $l^2>12$ with respect to the system's total energy E.

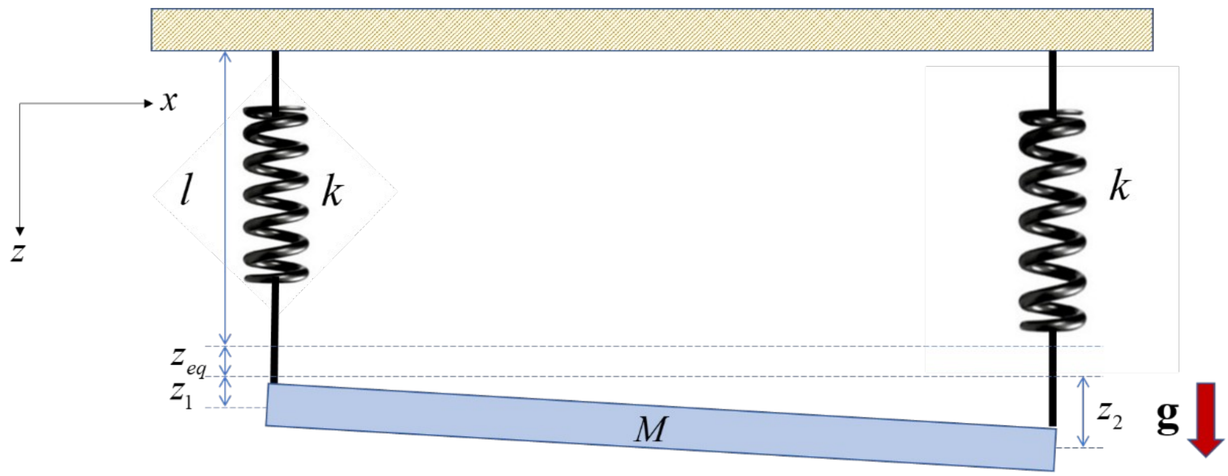
Problem 3 (30pts)



A cylinder with radius R and mass m is rolling down from the top of a ramp with mass M. The ramp is free to move without friction along the ground. The cylinder rolls without slipping down the ramp and always stays in contact with the surface of the ramp.

a) Find a set of generalized coordinates for the system and draw a diagram explicitly showing how they describe the system fully. Chose a set which will be convenient for the calculation of the constraint force using the Lagrange Multiplier method later in the problem.
b) Write down the two holonomic constraints corresponding to the condition for the cylinder staying on the surface of the ramp and its rolling without slipping.
c) Calculate the Lagrangian for the system using the set of generalized coordinates chosen in a).
d) Write down the equation of motion (ODEs) for the system by applying the Euler-Lagrangian equation.
e) Taking M=2m and using the Lagrangian Multiplier method, calculate the magnitude of the normal force needed to keep the cylinder on the surface of the ramp.

Problem 4 (30pts)



A bar of length d and mass M is suspended by two identical springs with spring constant k and unstretched length l on either end as show above. Consider small oscillations of the system in the x-z plane, such that the motion of the two ends of the bar will be (approximatelly) only in the z-direction.

a) Find z_{eq} when the system is in equilibrium under gravity.
b) Write out the kinetic energy T and the potential energy V in terms of the displacement variables, z_1 and z_2, for the two ends of the bar (see graph). Explicitly show that for small oscillations, both T and V will be in a quadratic form only except for a constant term for the potential which you can take as zero.
c) Using the quadratic form of T and V, find the eigenfrequencies of the two independent modes of the bar's small oscillation motion.
d) Find and describe the motion of the two normal modes of the bar corresponding to the two eigenfrequencies.