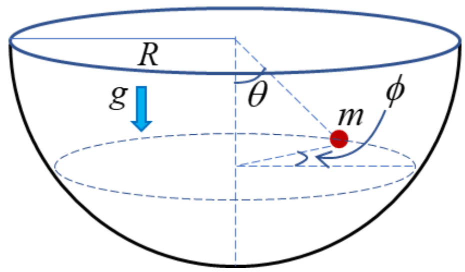
Classical Mechanics Qualifier (August 2021)

George Mason University

You are allowed to use your graduate textbook during the exam.
Four problems | Total 100 points

Problem 1 (20pts)

A particle with mass *m* moving frictionlessly inside a hemispherical bowl with radius *R* is described by the following Lagrangian *L*,



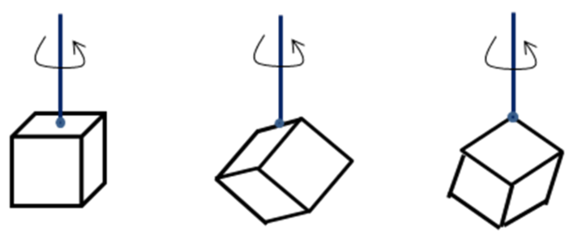
$$L = \frac{mR^2}{2} \phi^2 \sin^2\theta + \theta^2 \; + mgR\cos\theta$$

where ϕ and θ are respectively the polar angle and the azimuthal angle describing the location of the mass inside the bowl.

a) Find the two canonical momenta P_ϕ and P_θ .
b) Find the corresponding Hamiltonian for the system.
c) What are conserved quantities in this problem?
d) Find the equations of motions in the Hamiltonian formalism.

Problem 2 (20pts)

A torsion pendulum consists of a vertical wire attached to a mass *m* which may rotate about the vertical axis. *k* is the torque constant for the wire. Consider three such torsion pendula consisting of identical wires from which three identical homogeneous solid cubes are hung. All cubes have side *a* and mass *m*. One cube is hung from the middle of a face, one from midway along an edge, and one from a corner. The axis of rotation goes through the center of mass of the cube in all three cases.

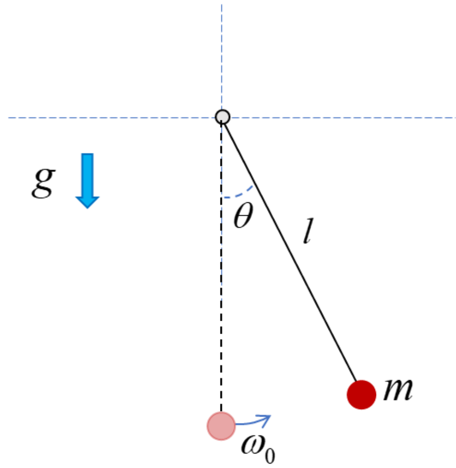


Case 1 Case 2 Case 3

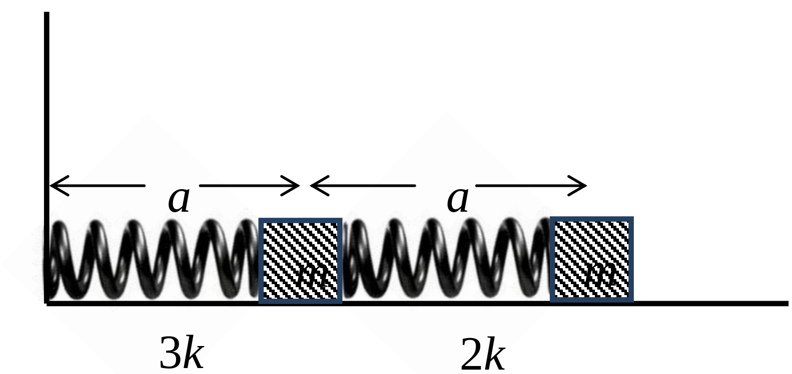
a) Calculate the moment of inertia tensor for a cube with side *a* and mass *m* with respect to its center of mass in all three cases.
b) What is the natural frequency of oscillations for these three torsion pendulums and which case will have the largest natural frequency?

Problem 3 (25pts)

A particle of mass *m* is suspended by a massless string of length *l*. It is initially just hanging without motion in a gravitational field of strength *g*. Then, it is struck by an impulsive horizontal force and starts moving with an angular velocity ω_0. If ω_0 is sufficiently small, the mass will simply move as a simple pendulum. If ω_0 is sufficiently large, the mass will rotate over the top above its point support. Use Lagrange Multiplier to determine the conditions in terms of ω_0, g, and *l* under which the string becomes slack as the mass moves up from its resting position.



Problem 4 (35pts)



Two equal masses *m* are attached to each other by a spring and the left mass is also attached to a fixed wall by a spring. The whole system lay flat on a frictionless surface and the spring-mass system is constrained to move only along one direction. The unstretched equilibrium length of the springs is *a*. The left spring has a spring constant of 3*k* and the right spring has a spring constant of 2*k*. The situation is illustrated in the diagram above.

a) Find the Lagrangian for the system.
b) Assuming small oscillations, find the resonant frequencies and normal modes for the system.
c) At *t*=0, the left mass is displaced from its equilibrium position by a value of –*b* to the left and the right mass is displaced from its equilibrium position by a value of +*b* to the right. The two masses are then released from this position with zero initial velocity. Find the positions of the two masses as a function of time *t*.