egrees-of-freedom system con-
ılinear oscillator with multiple
ar perturbations and the the-
e unforced dissipative system
ssing through all equilibrium
m possesses a global slow in-
on the slow invariant manifold
for small μ, is a regular per-
 oscillator approximates well
ant manifold.

chaotic motions of a singularly
Dissertation, Purdue University,

, John Wiley & Sons, New York.

on theory for ordinary differen-
31, 53-98.

a) Dynamics of nonlinear struc-
ant manifold approach, *ZAMP*
submitted).

Theory, Springer-Verlag, New

) Slow and fast invariant mani-
om structural dynamical system
urnal of Non-Linear Mechanics,

al mode vibrations of nonlinear
of Applied Mathematics 22 (3),

995b) Invariant manifolds and
r oscillators, *International Jour-*

CONTROL OF CHAOS: IMPACT OSCILLATORS AND TARGETING

ERNEST BARRETO[1,2]
University of Maryland, College Park, MD 20742.
FERNANDO CASAS
Departament de Matemàtiques, Universitat Jaume I, 12071-Castellón, Spain.
CELSO GREBOGI[2,3,4]
University of Maryland, College Park, MD 20742.
AND
ERIC J. KOSTELICH
Department of Mathematics, Arizona State University, Tempe, AZ 85287.

1. *Department of Physics.*
2. *Institute for Plasma Research.*
3. *Department of Mathematics*
4. *Institute for Physical Science and Technology.*

Abstract. We present two applications of chaos control techniques that can be of importance in mechanical systems. First, we apply chaos control to select a desired sequence of impacts in a map that captures the universal properties of impact oscillators near grazing. Next we describe a targeting method that can significantly reduce the chaotic transients that precede stabilization when these control methods are used.

1. Introduction

Recently, the application of chaos control techniques to physical systems has commanded increasing attention. In this work we describe the application of these methods to the impact oscillator, a mechanical system of great

D. H. van Campen (ed.), IUTAM Symposium on Interaction between Dynamics and Control in Advanced Mechanical Systems, 17–26.

importance. We also describe a targeting method that can improve the utility of control techniques when applied to higher dimensional chaotic systems.

An impact oscillator is a forced vibrating mechanical system which undergoes a sequence of contacts with motion-limiting constraints. The dynamics is therefore smooth motion, governed by a differential equation, interrupted by a series of non-smooth collisions. The collisions introduce nonlinearity into the system. Impact oscillators are used to model a variety of different systems arising in engineering (for example, moored ships colliding with fenders, forced mechanical systems with clearances such as rattling gears, and railway vehicles [1, 2])

Mathematically, impact oscillators constitute a subclass of dynamical systems that do not satisfy the usual smoothness assumptions. These discontinuities are responsible for new forms of behavior not found in smooth dynamical systems, particularly in the limit of low velocity or grazing impacts [1-8].

In engineering, systems are modelled and investigated in order to identify and avoid unacceptable responses. For impact systems, it is necessary to avoid high velocity impacts as these cause the greatest wear or damage to components. This can be accomplished by the well-known techniques of chaos control [9]. The flexibility provided by chaos allows us to select particular trajectories with a desirable sequence of impacts. This can be advantageous in many technological applications of impact oscillators.

In this work we apply the method of Ott, Grebogi and Yorke to control chaotic impacts in the Nordmark map [3, 10, 11] $(x_{n+1}, y_{n+1}) = F_\rho(x_n, y_n)$, where

$$F_\rho(x,y) = \begin{cases} (\alpha x + y + \rho, -\gamma x) & \text{for } \ x \le 0 \\ (-\sqrt{x} + y + \rho, -\gamma \tau^2 x) & \text{for } \ x > 0. \end{cases} \tag{1}$$

This is a piecewise continuously differentiable map that models the behavior of a sinusoidally forced linear oscillator experiencing impacts at a hard wall. It is obtained by expanding solutions of the system in the neighborhood of a grazing orbit [3], i.e., of an orbit that just touches the wall with zero velocity. The map captures the *universal* properties of the dynamics in the regime of low velocity impacts. The equivalence with the physical system is as follows: x_n and y_n are transformed coordinates in the position-velocity space $(\xi, \dot{\xi})$ evaluated at times $t_n = 2n\pi/\omega$, where ω is the frequency of the external forcing. The quantity τ^2 is the restitution coefficient of the impacts, and ρ is related to F_0, the amplitude of the external force. The parameters α and γ depend on the intrinsic properties of the oscillator such that the limit $\gamma \to 0$ corresponds to a large coefficient of friction, and $\gamma\tau^2 = 1$ gives the opposite limit of zero dissipation. For physical systems (with positive friction) we have [10-12]

$$0 < \gamma < 1, \qquad -2\sqrt{\gamma} < \alpha < 1 + \gamma. \tag{2}$$

The top expression in (1), valid for $x \leq 0$, governs the system if there is no impact between time t_n and t_{n+1}. Otherwise, $x > 0$ and the second expression applies. Thus, the effect of impacts in the system is modelled by a square root nonlinearity.

2. Control of the Impact Oscillator

The control technique of Ott, Grebogi and Yorke has the feature that it enables one to select a predetermined time-periodic behavior embedded in a chaotic attractor by making only *small* time-dependent perturbations to a set of accessible parameters of the system. The basic idea is as follows [9]. First one chooses a desirable unstable periodic orbit embedded in the chaotic attractor according to some set of performance criteria. Second, one defines a small region around the desired periodic orbit. A trajectory starting with almost any initial condition eventually falls into this small region by ergodicity. When this occurs, one applies perturbations to available control parameters so as to move the orbit onto the stable manifold of the desired unstable orbit. The flexibility of the method allows for the stabilization of different periodic orbits for the same set of nominal values of the parameter. This is possible because a chaotic attractor typically has embedded within it a large number of different unstable periodic orbits. We choose a single control parameter, ρ. This characterizes the strength of the driving. The grazing state corresponds to $\rho = 0$, and dynamics in the neighborhood of grazing is given for $|\rho| << 1$. Bifurcations occur as the parameter ρ is increased through $\rho = 0$ with γ and α held fixed.

By applying the OGY algorithm to the Nordmark map, one can stabilize periodic trajectories with an arbitrary number and an arbitrary distribution of impacts per period. This is so even if it is not possible to get analytic expressions for the position of the physical components. Also, the necessary information needed for applying control can be extracted purely from measured data [9, 13]. Here, for simplicity, we consider only maximal periodic orbits [10], i.e., periodic trajectories for which there is exactly one impact per period.

For systems with parameters in the region $4\gamma + \frac{1}{4} < \alpha < \frac{3}{2}\gamma + \frac{2}{3}$, windows of stable maximal periodic orbits are encountered as ρ is decreased from positive values [10, 11]. In particular, a window of period p is separated from the succeeding window, of period $p + 1$, by a band of chaos. There is an infinite cascade of such windows of decreasing width in ρ and increasing period, accumulating on $\rho = 0^+$. This is illustrated by the bifurcation

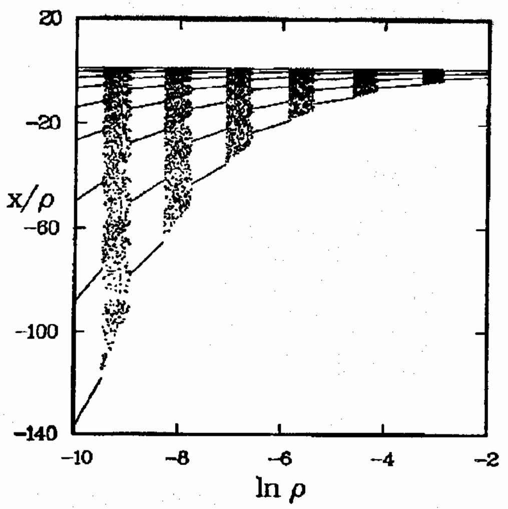


Figure 1. Bifurcation diagram for $(\gamma, \alpha) = (0.05, 0.65)$ and $\tau^2 = 1$ for small positive ρ values.

diagram of **Fig. 1**, obtained for $(\gamma, \alpha) = (0.05, 0.65)$ and $\tau^2 = 1$ for small positive ρ values. Here we can avoid the presence of chaotic impacts for $\rho > 0$ by applying control. As an example we take $\rho = \exp(-9.2)$, on the left band of chaos in Fig. 1. Here we have unstable maximal orbits up to period $M = 8$ embedded in the chaotic attractor.

Fig. 2 illustrates control of these periodic orbits. We plot the x-coordinate of a trajectory as a function of time. The parameter perturbations were programmed to successively control the seven different periodic orbits. Control for the $M = 2$ maximal orbit was turned on after 3000 free iterations. Each window was controlled for 500 iterations before switching to the next orbit. The figure shows that the time to achieve control is almost negligible in this case, with no apparent transients between switches. The maximum allowed parameter perturbation is $\delta = 10^{-4}$. Thus it is possible to convert chaotic impacts to controlled periodic orbits by applying only small perturbations $|\delta\rho| < 10^{-4}$ to the parameter ρ.

For parameters in the region $\frac{3}{2}\gamma + \frac{2}{3} < \alpha < 1 + \gamma$, there is an interval of ρ values occupied entirely by a chaotic attractor. As ρ increases from zero, this interval terminates at a stable maximal orbit of some period M_0 [10]. This attractor has embedded within it unstable maximal periodic orbits of increasing period as ρ approaches zero. An example of a bifurcation diagram for this case is shown in Fig. 3, obtained for $(\gamma, \alpha) = (0.15, 1)$ and $\tau^2 = 1$. Here we can control chaos by stabilizing any of the maximal orbits which are present for positive values of ρ. For $\rho = 0.05$, we have unstable

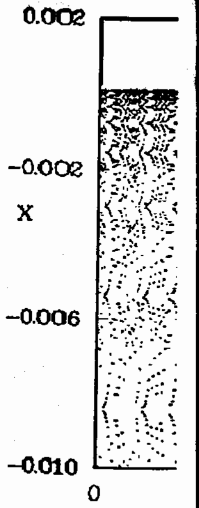


Figure 2. Successive control o
starting with period $M = 2$. The

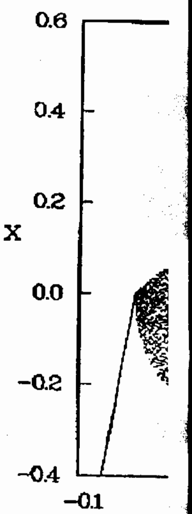
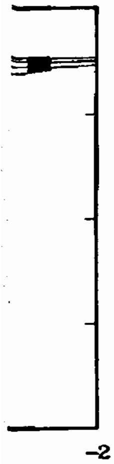


Figure 3. Bifurcatio



and $\tau^2 = 1$ for small positive ρ

0.65) and $\tau^2 = 1$ for small
ence of chaotic impacts for
take $\rho = \exp(-9.2)$, on the
table maximal orbits up to
or.
its. We plot the x-coordinate
eter perturbations were pro-
rent periodic orbits. Control
er 3000 free iterations. Each
switching to the next orbit.
ol is almost negligible in this
ches. The maximum allowed
possible to convert chaotic
ng only small perturbations

$1+\gamma$, there is an interval of
or. As ρ increases from zero,
bit of some period M_0 [10].
ble maximal periodic orbits
n example of a bifurcation
ed for $(\gamma, \alpha) = (0.15, 1)$ and
g any of the maximal orbits
$\rho = 0.05$, we have unstable

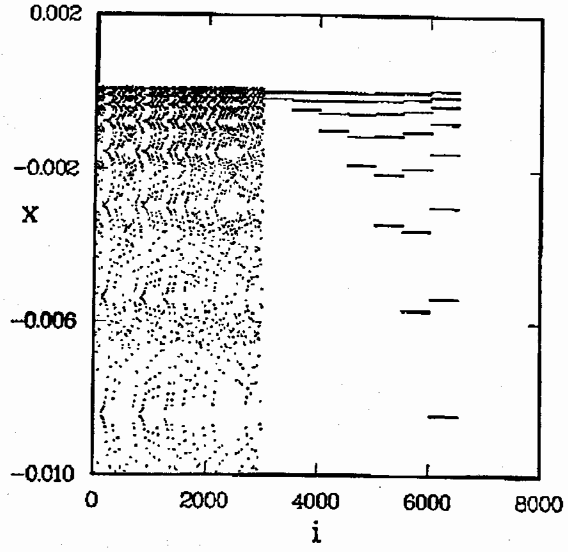


Figure 2. Successive control of unstable maximal periodic orbits for $\rho = \exp(-9.2)$, starting with period $M = 2$. The maximum parameter perturbation is $\delta = 10^{-4}$.

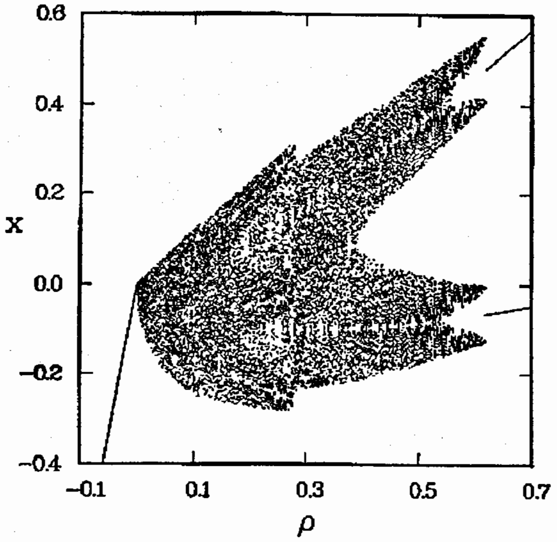


Figure 3. Bifurcation diagram for $\gamma = 0.15$, $\alpha = 1$ and $\tau^2 = 1$.

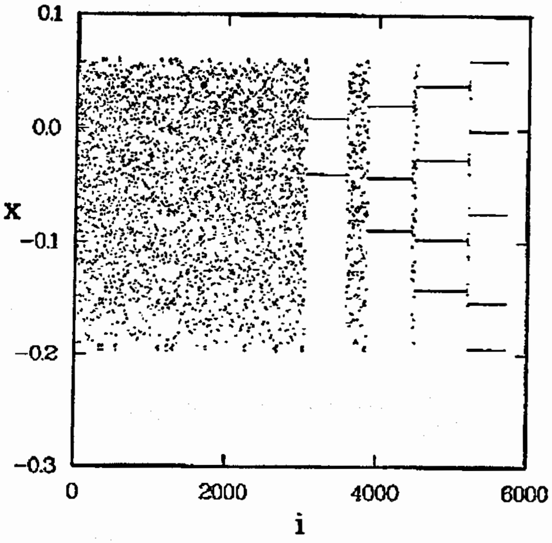


Figure 4. Successive control of the unstable maximal periodic orbits embedded in the chaotic attractor for $\rho = 0.05$, starting with period $M = 2$. The maximum parameter perturbation is $\delta = 10^{-3}$.

maximal orbits up to period $M = 5$. The control of these periodic orbits is accomplished as described above for Fig. 2.

3. Targeting of Periodic Orbits

The method described above relies on the natural ergodicity of chaotic dynamics to bring a trajectory into the vicinity of a desired unstable perioic orbit where it can be actively controlled. In applications involving higher dimensional systems, the times required for this to happen may be prohibitively long. For example, Romeiras *et al.* [9] have have applied the method to a four-dimensional map that describes a kicked double rotor [14], shown in Figure 5. They showed that control can be achieved by using only one control parameter (even when the attractor has two positive Lyapunov exponents). However, some unstable periodic orbits require several hundred thousand iterations before stabilization is achieved [15].

Targeting is a slightly different version of the control problem. We assume that we are given some initial condition on the attractor, and we wish to rapidly direct the resulting trajectory to a small region about some specified point on the attractor. Because of the inherent exponential sensitivity of chaotic time evolutions to perturbations, one expects that this can be accomplished using only small controlling adjustments of one or more

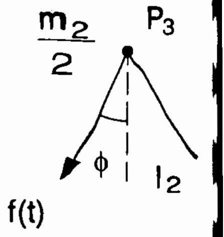
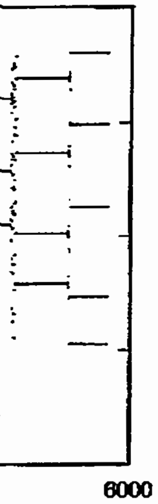


Figure 5. The Kicked Double Rc
ary point P_1. A second massless
is mounted at the end of the firs
are applied at an angle ϕ as sl
$(n+1)th$ kick is given by a four
$\mathbf{Y}_{n+1} = \mathbf{L}\mathbf{Y}_n + \mathbf{G}(\mathbf{X}_{n+1})$, where
$\mathbf{Y} = (\dot{\theta}_1, \dot{\theta}_2)^T$ are the correspondi
M and **L** are both constant matr
pivots and the moments of inerti
time n are $\rho_n = 9.0 + \Delta\rho_n$ and ϕ
take $l_1 = 1/\sqrt{2}$, and set all other

available system parameters
This was demonstrated
the case of a two-dimensior
laboratory experiment for w
dimensional map [17]. Kost
targeting procedure that ca
such as the double rotor ma
Because the dimension o
rameters chosen in Romeir.
between nearest neighbors i
as $N^{-1/2.8}$. This implies th
required to come within 10^{-}
control procedure described
10^{-4} of the target in less tha



periodic orbits embedded in the
= 2. The maximum parameter

ol of these periodic orbits is

ıtural ergodicity of chaotic
ːy of a desired unstable pe-
l. In applications involving
l for this to happen may be
l. [9] have have applied the
es a kicked double rotor [14],
n be achieved by using only
has two positive Lyapunov
bits require several hundred
ved [15].
he control problem. We as-
n on the attractor, and we
o a small region about some
inherent exponential sensi-
ns, one expects that this can
adjustments of one or more

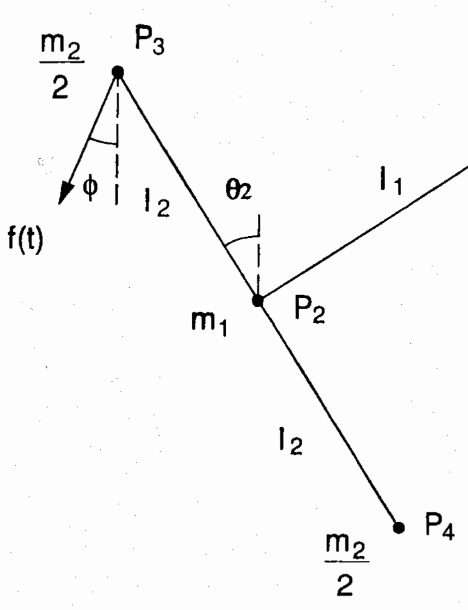


Figure 5. The Kicked Double Rotor. A massless rod of length l_1 pivots about the stationary point P_1. A second massless rod of length $2l_2$ is mounted on pivot P_2, which in turn is mounted at the end of the first rod. Periodic impulsive kicks $f(t) = \sum_{n=0}^{\infty} \rho_n \delta(t-n)$ are applied at an angle ϕ as shown. The state of the system immediately after the $(n+1)th$ kick is given by a four dimensional map of the form $\mathbf{X}_{n+1} = \mathbf{M}\mathbf{Y}_n + \mathbf{X}_n$ and $\mathbf{Y}_{n+1} = \mathbf{L}\mathbf{Y}_n + \mathbf{G}(\mathbf{X}_{n+1})$, where $\mathbf{X} = (\theta_1, \theta_2)^T$ are the two angular position coordinates, $\mathbf{Y} = (\dot{\theta}_1, \dot{\theta}_2)^T$ are the corresponding angular velocities, and $\mathbf{G}(\mathbf{X})$ is a nonlinear function. $\mathbf{M}$ and $\mathbf{L}$ are both constant matricies which involve the coefficients of friction at the two pivots and the moments of inertia of the rotor. Gravity is absent. Control parameters at time n are $\rho_n = 9.0 + \Delta\rho_n$ and $\phi_n = 0.0 + \Delta\phi_n$, with $|\Delta\rho|/\rho_0 \le 0.1$ and $|\Delta\phi| \le 0.5$. We take $l_1 = 1/\sqrt{2}$, and set all other parameters to 1. For further details, see Ref. [14].

available system parameters.

This was demonstrated theoretically and in numerical experiments for the case of a two-dimensional map by Shinbrot *et al.* [16], and also in a laboratory experiment for which the dynamics were approximated by a one dimensional map [17]. Kostelich *et al.* [18] developed an extension of the targeting procedure that can be applied to higher dimensional systems, such as the double rotor map.

Because the dimension of the double rotor attractor (for the set of parameters chosen in Romeiras, *et al.*) is about 2.8, the average distance between nearest neighbors in a subset of N points on the attractor scales as $N^{-1/2.8}$. This implies that, on average, 10^{11} iterations of the map are required to come within 10^{-4} of the target without the control. Since the control procedure described in [18] can steer the initial condition to within 10^{-4} of the target in less than 10^2 steps, the method can achieve the target about 10^9 times faster than the uncontrolled chaotic process.

The method works in two steps. First, information is learned about the system by observing a very long chaotic orbit, and constructing targeting trees as follows. The map is iterated from a random initial condition while keeping in memory a short history of the iterates encountered (for example, 10 consecutive points), until the orbit lands within a suitable tolerance distance of the target. This point, together with the recorded pre-iterates, comprise the *trunk path* of the tree, and are stored in memory. The map is then iterated again, still keeping track of a brief iterate history, until the orbit lands near any one of the points already in the tree. When this happens, a new path is added as a *branch*. Continuing in this way, a tree is built with a hierarchy of branches: the trunk path is level 1; level 2 branches are those that are rooted at some point in the trunk path; level 3 branches are rooted at a level 2 branch, and so on. The objective is to build a tree with enough branches such that a typical chaotic orbit lands near a point in the tree after a small number of iterations.

Once a sufficiently large targeting tree has been built, a chaotic orbit can be steered along the tree to the target. One applies small changes to available parameters to steer the orbit to the stable manifold of a point in the tree. (The *stable manifold* S associated with a typical point x is stable in the sense that $\|F^n(x) - F^n(y)\| \to 0$ as $n \to \infty$ whenever $y \in S$.) When the method is successful, the dynamics of the system carry the orbit of the perturbed point close to an orbit that leads directly to the target. Additional details on the method are given in [18].

The targeting algorithm can be combined with the OGY control method to provide a means to rapidly switch a given chaotic process between prespecified periodic orbits. That is, the targeting procedure can be used to steer a given initial condition on the attractor to a neighborhood of one of the periodic orbits, then the OGY control method can be used to stabilize the system near the periodic orbit. The combined method is discussed in [15], and the results of its application to the double rotor are shown in Figures 6 and 7 .

4. Conclusions

In summary, we have shown that chaotic dynamics in impact oscillators can be converted into motion on a desired periodic orbit by using only small parameter perturbations. In higher dimensional systems, it is possible to employ a targeting technique to reduce the length of the chaotic transients that precede stabilization. These results can be of importance in technological applications.

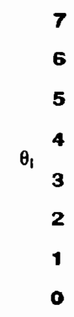


Figure 6. Graph illustratin
ordinate of the state is plott
orbit close to the desired UI
stabilized, and is not shown.

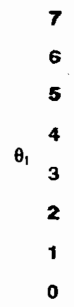


Figure 7. Improvements of
evident.

ACKNOWLEDGMENTS

This work was supportec
Computing). E. B. was
Consortium Fellowship u
tory. F. C. was supporte
Valenciana (Spain) and
Castelló 1994. E. K. is
Program in Applied Mat

References

1. Foale S. and Bishop, S.
tems, *Phil. Trans. R. S*
2. E. Slivsgaard and H. T

aotic process.
nation is learned about the
and constructing targeting
dom initial condition while
encountered (for example,
within a suitable tolerance
the recorded pre-iterates,
ored in memory. The map
brief iterate history, until
dy in the tree. When this
inuing in this way, a tree is
h is level 1; level 2 branches
unk path; level 3 branches
objective is to build a tree
ic orbit lands near a point

been built, a chaotic orbit
e applies small changes to
stable manifold of a point
with a typical point x is
$n \to \infty$ whenever $y \in S$.)
the system carry the orbit
ads directly to the target.
8].

h the OGY control method
aotic process between pre-
procedure can be used to
a neighborhood of one of
od can be used to stabilize
ed method is discussed in
double rotor are shown in

mics in impact oscillators
riodic orbit by using only
nsional systems, it is pos-
the length of the chaotic
lts can be of importance in

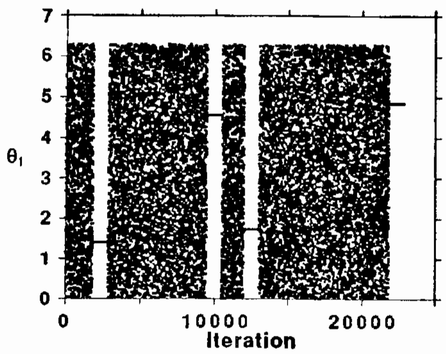


Figure 6. Graph illustrating switching between five different fixed points. The θ_1 coordinate of the state is plotted versus iteration. Here we rely on ergodicity to bring the orbit close to the desired UPO. The fifth fixed point required 153,485 iterations to be stabilized, and is not shown.

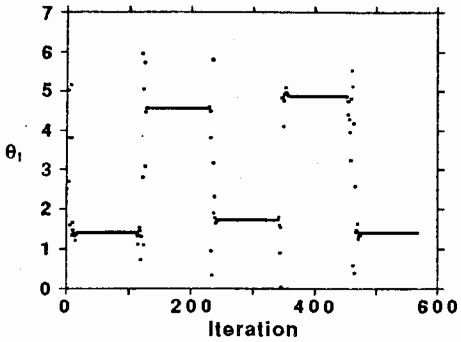


Figure 7. Improvements of up to four orders of magnitude in the switching times is evident.

ACKNOWLEDGMENTS

This work was supported by the Department of Energy (Office of Scientific Computing). E. B. was supported in part by a National Physical Science Consortium Fellowship under the sponsorship of Argonne National Laboratory. F. C. was supported by the Conselleria de Educació de la Generalitat Valenciana (Spain) and by the collaboration program UJI-Fundació Caixa Castelló 1994. E. K. is supported in part by the Department of Energy Program in Applied Mathematics, grant number DE-FG03-94ER25213.

References

1. Foale S. and Bishop, S.R. (1992) Dynamical complexities of forced impacting systems, *Phil. Trans. R. Soc. Lond.*, **A338**, pp. 547–556.
2. E. Slivsgaard and H. True (1994) Chaos in railway-vehicle dynamics, in J.M.T.

Thompson and S.R. Bishop (eds.), *Nonlinearity and Chaos in Engineering Dynamics*, John Wiley and Sons Ltd, England, pp. 183–192.

3. Nordmark, A.B. (1991) Non-periodic motion caused by grazing incidence in an impact oscillator, *J. Sound Vib.*, **145**, pp. 279–297.
4. Budd, C. and Dux, F. (1994) Intermittency in impact oscillators close to resonance, *Nonlinearity*, **7**, pp. 1191–1224.
5. Shaw, S.W. and Holmes, P.J. (1983) A periodically forced piecewise linear oscillator, *J. Sound Vib.* **90**, pp. 129–155.
6. Shaw, S.W. (1985) The dynamics of a harmonically excited system having rigid amplitude constraints, *ASME J. Appl. Mech.*, **52**, pp. 453–464.
7. Nusse, H.E. and Yorke, J.A. (1992) Border-collision bifurcations including 'period two to period three' for piecewise smooth systems, *Physica D*, **57**, pp. 39–57.
8. Budd, C., Dux, F. and Cliffe, A. (1995) The effect of frequency and clearance variations on single degree of freedom impact oscillators, *J. Sound Vib.*, **184**, pp. 475–502.
9. E. Ott, C. Grebogi and J.A. Yorke (1990) Controlling chaos, *Phys. Rev. Lett.*, **64**, pp. 1196–1199; F.J. Romeiras, C. Grebogi, E. Ott and W.P. Dayawansa (1992) Controlling chaotic dynamical systems, *Physica D* **58**, pp. 165–192; E. Barreto and C. Grebogi (1995) Multiparameter control of chaos, *Phys. Rev. E*, **52**, pp. 3553–3557 . See also E. Barreto, Y.C. Lai and C. Grebogi (1996), Controlling Chaos with applications to Mechanical Systems, in F.C. Moon (ed.), *Nonlinear Dynamics of Material Processing and Manufacturing*, John Wiley and Sons, Inc., New York, to be published.
10. Chin, W., Ott, E., Nusse, H.E. and Grebogi, C. (1994) Grazing bifurcations in impact oscillators, *Phys. Rev. E*, **50**, pp. 4427–4444.
11. Chin, W., Ott, E., Nusse, H.E. and Grebogi, C. (1995) Universal behavior of impact oscillators near grazing incidence, *Phys. Lett. A*, **201**, pp. 197–204.
12. Casas, F., Chin, W., Grebogi, G. and Ott, E. (1995) Universal grazing bifurcations in impact oscillators, *submitted for publication.*
13. U. Dressler and G. Nitsche (1992) Controlling chaos using time delay coordinates, *Phys. Rev. Lett*, **68**, pp. 1–4; P. So and E. Ott (1995) Controlling chaos using time delay coordinates via stabilization of periodic orbits, *Phys. Rev. E*, **51**, pp. 2955–2962.
14. E.J. Kostelich, C. Grebogi, E. Ott and J.A. Yorke (1987) Multi-dimensioned intertwined basin boundaries: basin structure of the kicked double rotor, *Physica D*, **25**, pp. 347–360; C. Grebogi, E.J. Kostelich, E. Ott and J.A. Yorke (1986) Multi-dimensioned intertwined basin boundaries and the kicked double rotor, *Phys. Lett. A*, **118**, pp. 448–452 and errata(1987), **120**, pp. 497.
15. E. Barreto, E.J. Kostelich, C. Grebogi, E. Ott and J.A. Yorke (1995) Efficient switching between controlled unstable periodic orbits in higher dimensional chaotic systems, *Phys. Rev. E*, **51**, pp 4169–4172.
16. T. Shinbrot, E. Ott, C. Grebogi and J.A. Yorke (1990) Using chaos to direct trajectory to targets, *Phys. Rev. Lett.*, **65**, pp. 3215–3218.
17. T. Shinbrot, W. Ditto, C. Grebogi, E. Ott, M. Spano and J.A. Yorke (1992) Using the sensitive dependence of chaos (the 'butterfly effect') to direct trajectories in an experimental chaotic system, *Phys. Rev. Lett.*, **68**, pp. 2863–2866.
18. E.J. Kostelich, C. Grebogi, E. Ott, and J.A. Yorke (1993) Higher-dimensional targeting, *Phys. Rev. E*, **47**, p. 305–310.

HILL'S PROBLEM A

V.V.BELETSKY
The Keldysh Inst
Sciences,Miusska
e-mail: beletsky@

Projects exist for expedi
using jumping robots. These p
under the simultaneous action
satellite (Phobos or Daimos) t
surface.

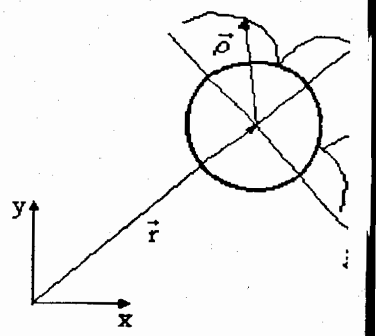


Fig. 1

Its origin coincides with the m
this co-ordinate system, the sa
face towards the planet.

The satellite is considere
the vicinity of the satellite is go

$\xi'' +$

$\eta'' -$

These equations are su
satellite surface. The impacts ar

D. H. van Campen (ed.), IUTAM Sympos
Mechanical Systems, 27–34.