SYNCHRONY IN GLOBALLY COUPLED CHAOTIC, PERIODIC, AND MIXED ENSEMBLES OF DYNAMICAL UNITS

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Abstract

The onset of collective synchronous behavior in globally coupled ensembles of oscillators is discussed. We present a formalism that is applicable to general ensembles of heterogeneous, continuous time dynamical units that, when uncoupled, are chaotic, periodic, or a mixture of both. A discussion of convergence issues, important for the proper implementation of our method, is included. Our work leads to a quantitative prediction for the critical coupling value at the onset of collective synchrony and for the growth rate of the resulting coherent state.

Systems that consist of many coupled heterogeneous dynamical units are of great interest in a wide variety of situations. The simplest such situation is where the coupling is *global* in the sense that each unit is coupled to all other units. Past work has concentrated on the case where the dynamics of the uncoupled units is periodic with a spread in the oscillator frequencies [1]. In that case, a typical behavior is that, for sufficiently low coupling, the individual units oscillate incoherently, but that, as the magnitude of the coupling increases through a critical value, there is a transition to coherent system dynamics in which a group of oscillators in the ensemble of units becomes locked in frequency and phase. Possible applications include synchrony in chirping crickets [2], flashing fireflies [3], Josephson junction

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arrays [4], semiconductor laser arrays [5], and cardiac pacemakers cells [6]. More recently, the similar question of what happens when the uncoupled dynamics of the individual units is chaotic has been addressed [7, 8, 9] (see also the recent experimental study of globally coupled chaotic electrochemical oscillators [10]). In this contribution we discuss a formalism that is capable of treating the onset of synchronism of a general system of globally coupled, heterogeneous, continuous-time dynamical units. No *a priori* assumption regarding the uncoupled dynamics of the individual units is made. Thus, one can consider chaotic or periodic dynamics of the uncoupled units, including the case where both types of units are present in the same system.

In order to illustrate the context for the subsequent presentation of our analysis, we first discuss some numerical examples. We consider a system consisting of an ensemble of units in which each one of the units, when uncoupled from the others, obeys the Lorenz equations: $dx^{(1)}/dt = \sigma(x^{(2)} - x^{(1)})$; $dx^{(2)}/dt = rx^{(1)} - x^{(2)} - x^{(1)}x^{(3)}$; $dx^{(3)}/dt = -bx^{(3)} + x^{(1)}x^{(2)}$. We denote the state variables for unit i of the ensemble by $(x_i^{(1)}, x_i^{(2)}, x_i^{(3)})$ and we couple the units by adding a global coupling term $(K/N)\sum_{i=1}^{N} x_i^{(1)}(t)$ to the right side of the $dx_i^{(1)}/dt$ equation. Here, N is the number of units in the ensemble and is presumed to be large ($N >> 1$); in the subsequent analysis, the limit $N \to \infty$ is employed. In addition, the parameter r will be different for each ensemble member, and we denote its value for unit i by r_i. For our numerical examples, the values of r_i are taken to be uniformly distributed in an interval $[r_-, r_+]$ and $N > 10^4$. We numerically examine the following three ensembles:

Case (a): chaotic ensemble, $[r_-, r_+] = [28, 52]$.

Case (b): periodic ensemble, $[r_-, r_+] = [150, 165]$.

Case (c): mixed ensemble, $[r_-, r_+] = [167, 202]$.

In case (a), numerical computations show that the uncoupled Lorenz equations yield chaotic solutions with no discernable windows of periodicity in the range $[r_-, r_+]$. In case (b), the uncoupled solutions are periodic, but there is a pitchfork bifurcation at a value $r = r_p$, $r_- < r_p < r_+$ such that as r decreases through r_p, a symmetric periodic orbit where the symmetry is $(x_i^{(1)}, x_i^{(2)}, x_i^{(3)}) \to (-x_i^{(1)}, -x_i^{(2)}, x_i^{(3)})$ bifurcates to two individually asymmetric orbits. In case (c), the ensemble is substantially chaotic but with a large window of periodicity in $[r_-, r_+]$. Figures viii (a-c) give numerical results for cases (a-c). These figures show an order parameter $\bar{x}_T$ versus the coupling coefficient k, where

$$\bar{x}_T = \left\{ \frac{1}{T} \int_t^{t+T} \left(\frac{1}{N} \sum_{i=1}^{N} x_i^{(1)}(t') \right)^2 dt' \right\}^{1/2}$$

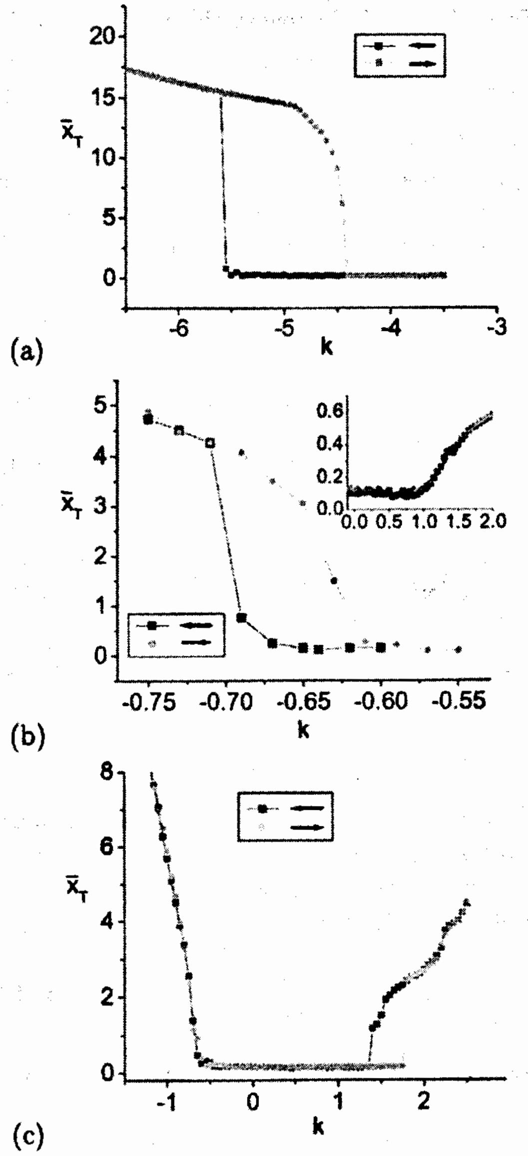

Figure 1. $\bar{x}_T$ versus k for cases (a), (b), and (c).

and $\bar{x}_T$ characterizes the degree of coherent motion of the system. The times t and T are chosen large enough that the system settles into its time-asymptotic dynamics and $\bar{x}_T$ is essentially independent of T. Note that, if the individual units behave incoherently, the sum is close to zero by the symmetry of the uncoupled Lorenz equations. For case (a), there is a subcritical bifurcation as k decreases through $k_a = -5.6$. For case (b), there is a supercritical Hopf bifurcation as k increases through $k_b^{(+)} \approx 1$ and a subcritical Hopf bifurcation as k decreases through $k^{(-)} \approx -0.68$. For case (c), there is a subcritical Hopf bifurcation at $k_c^{(+)} \approx 1.8$, and a supercritical Hopf bifurcation at $K_c^{(-)} \approx -0.7$. It is our goal to obtain a theory for these critical k values at the onset of coherence. These mark the onset of instability of the incoherent state, and we are also interested in the exponential growth rates of these instabilities.

We now present our analysis, treating the simplest case (generalizations will be given elsewhere [11]). We consider dynamical systems of the form

$$d\mathbf{x}_i(t)/dt = \mathbf{G}(\mathbf{x}_i(t), \mathbf{\Omega}_i) + \mathbf{K}(\langle\langle \mathbf{x} \rangle\rangle_* - \langle\langle \mathbf{x}(t) \rangle\rangle), \tag{1}$$

where $\mathbf{x}_i = (x_i^{(1)}, x_i^{(2)}, \ldots, x_i^{(q)})^T$ is a q-dimensional vector; $\mathbf{G}$ is a q-dimensional vector function; $\mathbf{K}$ is a constant $q \times q$ coupling matrix; $i = 1, 2, \cdots, N$; $\langle\langle \mathbf{x}(t) \rangle\rangle$ is the instantaneous average

$$\langle\langle \mathbf{x}(t) \rangle\rangle = \lim_{N\to\infty} N^{-1} \sum_i \langle \mathbf{x}_i(t) \rangle, \tag{2}$$

and, for each i, $\langle \mathbf{x}_i \rangle$ is the average of $\mathbf{x}_i$ over an infinite number of initial conditions $\mathbf{x}_i(0)$ distributed on the attractor of the ith uncoupled system,

$$d\mathbf{x}_i/dt = \mathbf{G}(\mathbf{x}_i, \mathbf{\Omega}_i). \tag{3}$$

$\mathbf{\Omega}_i$ is a parameter vector specifying the uncoupled ($\mathbf{K} = 0$) dynamics, and $\langle\langle \mathbf{x} \rangle\rangle_*$ is the *natural measure* [12] and i average of the state of the uncoupled system. That is, to compute $\langle\langle \mathbf{x} \rangle\rangle_*$, we set $\mathbf{K} = 0$, compute the solutions to Eq. (3), and obtain $\langle\langle \mathbf{x} \rangle\rangle_*$ from

$$\langle\langle \mathbf{x} \rangle\rangle_* = \lim_{N\to\infty} N^{-1} \sum_i \left[\lim_{\tau_0\to\infty} \tau_0^{-1} \int_0^{\tau_0} \mathbf{x}_i(t)dt \right]. \tag{4}$$

In what follows we assume that the $\mathbf{\Omega}_i$ are randomly chosen from a smooth probability density function $\rho(\mathbf{\Omega})$. Thus, an alternate means of expressing (4) is

$$\langle\langle \mathbf{x} \rangle\rangle_* = \int \mathbf{x}\rho(\mathbf{\Omega}) d\mu_{\mathbf{\Omega}} d\mathbf{\Omega}, \tag{5}$$

where $\mu_{\mathbf{\Omega}}$ is the natural invariant measure for the system $d\mathbf{x}/dt = \mathbf{G}(\mathbf{x}, \mathbf{\Omega})$. By construction, $\langle\langle \mathbf{x} \rangle\rangle = \langle\langle \mathbf{x} \rangle\rangle_*$ is a solution of the globally coupled system (1). We call this solution the "incoherent state" because the coupling term cancels and the individual oscillators do not affect each other. The question we address is whether the incoherent state is stable. In particular, as a system parameter such as the coupling strength varies, the onset of instability of the incoherent state signals the start of coherent, synchronous behavior of the ensemble.

To perform the stability analysis, we assume that the system is in the incoherent state, so that at any fixed time t, and for each i, $\mathbf{x}_i(t)$ is distributed according to the natural measure. We then perturb the orbits $\mathbf{x}_i(t) \to \mathbf{x}_i(t) + \delta\mathbf{x}_i(t)$, where $\delta\mathbf{x}_i(t)$ is an infinitesimal perturbation:

$$d\delta\mathbf{x}_i/dt = \mathbf{DG}(\mathbf{x}_i(t), \mathbf{\Omega}_i)\delta\mathbf{x}_i - \mathbf{K}\langle\langle\delta\mathbf{x}_i\rangle\rangle \tag{6}$$

where

$$\mathbf{DG}(\mathbf{x}_i(t), \mathbf{\Omega}_i)\delta\mathbf{x}_i = \delta\mathbf{x}_i \cdot \frac{\partial}{\partial\mathbf{x}_i}\mathbf{G}(\mathbf{x}_i(t), \mathbf{\Omega}_i).$$

Introducing the fundamental matrix $\mathbf{M}_i(t)$ for system (6),

$$d\mathbf{M}_i/dt = \mathbf{DG} \cdot \mathbf{M}_i, \tag{7}$$

where $\mathbf{M}_i(0) \equiv \mathbb{1}$, we can write the solution of Eq. (6) as

$$\delta\mathbf{x}_i(t) = -\int_{-\infty}^{t} \mathbf{M}_i(t)\mathbf{M}_i^{-1}(\tau)\mathbf{K}\langle\langle\delta\mathbf{x}\rangle\rangle_\tau d\tau, \tag{8}$$

where we use the notation $\langle\langle\delta\mathbf{x}\rangle\rangle_\tau$ to signify that $\langle\langle\delta\mathbf{x}\rangle\rangle$ is evaluated at time τ. Note that, through Eq. (7), $\mathbf{M}_i$ depends on the unperturbed orbits $\mathbf{x}_i(t)$ of the uncoupled nonlinear system (3), which are determined by their initial conditions $\mathbf{x}_i(0)$ (distributed according to the natural measure).

Assuming that $\langle\langle\delta\mathbf{x}\rangle\rangle$ evolves exponentially in time (i.e., $\langle\langle\delta\mathbf{x}\rangle\rangle = \mathbf{\Delta}e^{st}$), Eq. (8) yields

$$\{\mathbb{1} + \tilde{\mathbf{M}}(s)\mathbf{K}\}\mathbf{\Delta} = 0, \tag{9}$$

where s is complex, and

$$\tilde{\mathbf{M}}(s) = \left\langle\left\langle\int_{-\infty}^{t} e^{-s(t-\tau)}\mathbf{M}_i(t)\mathbf{M}_i^{-1}(\tau)d\tau\right\rangle\right\rangle_* . \tag{10}$$

Thus the dispersion function determining s is

$$D(s) = \det\{\mathbb{1} + \tilde{\mathbf{M}}(s)\mathbf{K}\} = 0. \tag{11}$$

In order for Eqs. (9) and (11) to make sense, the right side of Eq. (10) must be independent of time. As written, it may not be clear that this is so. We now demonstrate this, and express $\tilde{\mathbf{M}}(s)$ in a more convenient form. Writing the dependence of $\mathbf{M}_i$ in Eq. (10) on the initial condition explicitly, we have from the definition of $\mathbf{M}_i$, $\mathbf{M}_i(t, \mathbf{x}_i(0))\mathbf{M}_i^{-1}(\tau, \mathbf{x}_i(0)) = \mathbf{M}_i(t-\tau, \mathbf{x}_i(\tau)) = \mathbf{M}_i(T, \mathbf{x}_i(t-T))$, where $T = t-\tau$. Using this in Eq. (10), we have

$$\tilde{\mathbf{M}}(s) = \langle\langle\int_0^{\infty} e^{-sT}\mathbf{M}_i(T, \mathbf{x}_i(t-T)dT\rangle\rangle_* .$$

Note that our solution requires that the integral in the above converge. Since the growth of $\mathbf{M}_i$ with increasing T is dominated by h_i, the largest Lyapunov exponent for the orbit $\mathbf{x}_i$, we require $\mathrm{Re}(s) > \Gamma$, where $\Gamma = \max_{\mathbf{x}_i,\mathbf{\Omega}_i}(h_i)$. In contrast with the chaotic case where $\Gamma > 0$, an ensemble of periodic attractors has $\Gamma = 0$ (for an attracting periodic orbit $h_i = 0$ corresponds to orbit perturbations along the flow). With the condition $Re(s) > \Gamma$, the integral converges exponentially and uniformly in the quantities over which we average. Thus we can interchange the integration and the average,

$$\tilde{\mathbf{M}}(s) = \int_0^\infty e^{-sT} \langle\langle \mathbf{M}_i(T, \mathbf{x}_i(t-T)) \rangle\rangle_* dT. \tag{12}$$

In Eq. (12) the only dependence on t is through the initial condition $\mathbf{x}_i(T-t)$. However, since the quantity within angle brackets includes not only an average over i, but also an average over initial conditions with respect to the natural measure of each uncoupled attractor i, the time invariance of the natural measure ensures that Eq. (12) is independent of t. In particular, invariance of a measure means that if an infinite cloud of initial conditions $\mathbf{x}_i(0)$ is distributed on uncoupled attractor i at $t = 0$ according to its natural invariant measure, then the distribution of the orbits, as they evolve to any time t via the uncoupled dynamics (Eq. (3)), continues to give the same distribution as at time $t = 0$. Hence, although $\mathbf{M}_i(T, \mathbf{x}_i(t-T))$ depends on t, when we average over initial conditions, the result $\langle \mathbf{M}_i(T, \mathbf{x}_i(t-T)) \rangle_*$ is independent of t for each i. Thus we drop the dependence of $\langle\langle \mathbf{M}_i \rangle\rangle_*$ on the initial values of the $\mathbf{x}_i$ and write

$$\tilde{\mathbf{M}}(s) = \int_0^\infty e^{-sT} \langle\langle \mathbf{M}(T) \rangle\rangle_* dT, \tag{13}$$

where, for convenience we have also dropped the subscript i. Thus $\tilde{\mathbf{M}}$ is the Laplace transform of $\langle\langle \mathbf{M} \rangle\rangle_*$. As we will see, this result for $\tilde{\mathbf{M}}(s)$ can be analytically continued into $Re(s) < \Gamma$.

Note that $\tilde{\mathbf{M}}(s)$ depends only on the solution of the linearized *uncoupled* system (Eq. (7)). Hence the utility of the dispersion function $D(s)$ given by Eq. (11) is that it determines the linearized dynamics of the globally coupled system in terms of those of the individual uncoupled systems.

Consider the jth column of $\langle\langle \mathbf{M}(t) \rangle\rangle_*$, which we denote $[\langle\langle \mathbf{M}(t) \rangle\rangle_*]_j$. According to our definition of $\mathbf{M}_i$ given by Eq. (7), we can interpret $[\langle\langle \mathbf{M}(t) \rangle\rangle_*]_j$ as follows. Assume that for each of the uncoupled systems i in Eq. (3), we have a cloud of an infinite number of initial conditions sprinkled randomly according to the natural measure on the uncoupled attractor. Then, at $t = 0$, we apply an equal infinitesimal displacement δ_j in the direction j to each orbit in the cloud. That is, we replace $\mathbf{x}_i(0)$ by $\mathbf{x}_i(0) + \delta_j \mathbf{a}_j$, where $\mathbf{a}_j$ is a unit vector in $\mathbf{x}$-space in the direction j. Since the particle cloud is displaced from the attractor, it relaxes back to the attractor as time evolves. The quantity $[\langle\langle \mathbf{M} \rangle\rangle_*]_j \delta_j$ gives the time evolution of the i-averaged perturbation of the centroid of the cloud as it evolves back to the attractor and redistributes itself on the attractor.

We now argue that $\langle\langle \mathbf{M} \rangle\rangle_*$ decays to zero exponentially with increasing time. We consider the general case where the support of the smooth density $\rho(\mathbf{\Omega})$ contains open regions of $\mathbf{\Omega}$ for which the dynamical system (3) has attracting periodic orbits as well as a positive measure of $\mathbf{\Omega}$ on which Eq. (3) has chaotic orbits. Numerical experiments on chaotic attractors (including structurally unstable attractors) generally show that they are strongly mixing; i.e., a cloud of many particles rapidly arranges itself on the attractor according to the natural measure. Thus, for each $\mathbf{\Omega}_i$ giving a chaotic attractor, it is reasonable to assume that the average of $\mathbf{M}_i$ over initial conditions $\mathbf{x}_i(0)$, denoted $\langle \mathbf{M}_i \rangle_*$, decays exponentially. For a periodic attractor, however, $\langle \mathbf{M}_i \rangle_*$ does not decay: a distribution of orbits along a limit cycle comes to the same distribution after one period, and this repeats forever. Thus, if the distribution on the limit cycle was noninvariant, it remains noninvariant and oscillates forever at the period of the periodic orbit. On the other hand, periodic orbits exist in open regions of $\mathbf{\Omega}$, and, when we average over $\mathbf{\Omega}$, there is the possibility that with increasing time cancellation causing decay occurs via the process of "phase mixing". For this case we appeal to an example. In particular, the explicit computation of $\langle \mathbf{M}_i \rangle_*$ for a simple model limit cycle ensemble is given in Ref. [11]. The result is

$$\langle \mathbf{M}_i \rangle_* = \frac{1}{2} \begin{bmatrix} \cos \Omega_i t & -\sin \Omega_i t \\ \sin \Omega_i t & \cos \Omega_i t \end{bmatrix},$$

and indeed this oscillates and does not decay to zero. However, if we average over the oscillator distribution $\rho(\Omega)$ we obtain

$$\langle\langle \tilde{\mathbf{M}} \rangle\rangle_* = \frac{1}{2} \begin{bmatrix} c(t) & -s(t) \\ s(t) & c(t) \end{bmatrix},$$

where $c(t) = \int \rho(\Omega) cos \Omega t d\Omega$ and $s(t) = \int \rho(\Omega) \sin \Omega t d\Omega$. For any analytic $\rho(\Omega)$ these integrals decay exponentially with time. Thus, based on these considerations of chaotic and periodic attractors, we see that for sufficiently smooth $\rho(\mathbf{\Omega})$, there is reason to believe that $\langle\langle \mathbf{M} \rangle\rangle_*$, the average of $\mathbf{M}_i$ over $\mathbf{x}_i(0)$ and over $\mathbf{\Omega}_i$, decays exponentially to zero with increasing time. Conjecturing this decay to be exponential, $\left\| \langle\langle \mathbf{M}(t) \rangle\rangle_* \right\| < \kappa e^{-\gamma t}$ for positive constants κ and γ, we see that the integral in Eq. (13) converges for $Re(s) > -\gamma$. This conjecture is supported by our numerical results. Thus, while Eq. (13) was derived under the assumption $Re(s) > \Gamma > 0$, using analytic continuation, we can regard Eq. (13) as valid for $Re(s) > -\gamma$. Note that, for our purposes, it suffices to require only that $\|\langle\langle \mathbf{M}(t)\rangle\rangle_* \|$ be bounded, rather than that it decay exponentially. Boundedness corresponds to $\gamma = 0$, which is enough for us, since, as soon as instability occurs, the relevant root of $D(s)$ has $Re(s) > 0$.

In order to apply Eq. (11), to a given situation, it is necessary to numerically approximate the matrix $\tilde{\mathbf{M}}(s)$. To do this we consider two possible candidate approaches.

Approach (i): First approximate the natural measure on each attractor i by a large finite number of orbits initially distributed according to the natural measure. For each initial condition, obtain $\mathbf{x}_i(t)$ from Eq. (3). Use these solutions in **DG** and solve Eq. (7). Then average over the natural measure and i to obtain $\langle\langle \mathbf{M}(t)\rangle\rangle_*$, and do the Laplace transform (Eq. (13).

Approach (ii): Since $\langle\langle \mathbf{M}\rangle\rangle_*$ is the response to an impulse (i.e., the sudden displacement of each orbit), its Laplace transform multiplied by $\exp(st)$, namely $M(s)\exp(st)$, is the response to the drive $\exp(st)\mathbb{1}$ added to the right side of Eq. (6). This suggests the following numerical prodedure for finding M(s). Solve

$$\frac{d\tilde{\mathbf{x}}_i^{(c,s)}}{dt} = \mathbf{G}(\tilde{\mathbf{x}}_i^{(c,s)}, \mathbf{\Omega}_i) + \mathbf{\Delta}_j \mathbf{a}_j e^{\sigma t} \begin{cases} \cos\omega t \\ \sin\omega t, \end{cases} \tag{14}$$

where $s = \sigma - i\omega$, and $\mathbf{a}_j$ is a unit vector in the direction j. For large t, but $\delta_j \exp(\sigma t)$ still small throughout the time interval $(0, t)$, we can regard the average response as approximately linear. Thus, the jth column of $\tilde{\mathbf{M}}(s)$ is

$$[\langle\langle \mathbf{M}(t)\rangle\rangle_*]_j = \Delta_j^{-1}(\langle\langle \mathbf{x}\rangle\rangle_* - \langle\langle \mathbf{x}'\rangle\rangle), \tag{15}$$

where $\tilde{\mathbf{x}}_i = \tilde{\mathbf{x}}_i^{(c)} - \tilde{\mathbf{x}}_i^{(s)}$. Numerically, $\langle\langle \tilde{\mathbf{x}}\rangle\rangle$ can be approximated using a large finite number of orbits. In Ref. [8], a technique equivalent to this with s taken to be imaginary ($s = -i\omega$) was used to obtain marginal stability [13].

For the coupling we have chosen to use for our numerical experiments, only $\tilde{M}_{11}(s)$ is nonzero, and thus Eq. (11) reduces to

$$1 + \tilde{M}_{11}(-i\omega)k = 0 \tag{16}$$

where we have set $s = -i\omega$. Solving $\text{Im}[\tilde{M}_{11}(-i\omega)] = 0$ yields roots $\omega = \omega^*$, which, when reinserted into Eq. (16) yield possible values $k = k^* = -[\tilde{M}_{11}(-i\omega^*)]^{-1}$ for the critical coupling strengths. To determine which of the possibly multiple roots ω^* are relevant, we envision that as k is increased or decreased from zero, a critical coupling value is encountered at which the incoherent state first becomes unstable. Hence we are interested in the roots $\omega^*_{a,b}$ corresponding to the smallest $|k^*|$ for k^* both negative

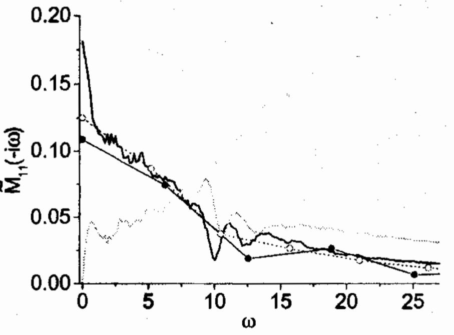

Figure 2. $\tilde{M}_{11}(-i\omega)$ versus ω for case (a). The solid black line is $\mathrm{Re}(\tilde{M}_{11})$ from approach (ii); the solid grey line is $\mathrm{Im}(\tilde{M}_{11})$ from approach (ii); and the dashed line is $\mathrm{Re}(\tilde{M}_{11})$ from approach (i).

($k^* = -|k_b^*|$) and positive ($k^* = k_b^* > 0$). Growth rates and frequency shifts from ω^* can also be simply obtained for k near k^* by setting $k = k^* + \delta k$, $s = -i(\omega^* + \delta\omega) + \gamma$ and expanding Eq. (11) for small δk, $\delta\omega$, and γ; e.g.,

$$\gamma = -\frac{\delta k}{(k^*)^2} \frac{\partial Im[\tilde{M}_{11}(-i\omega)]/\partial\omega}{|\partial\tilde{M}_{11}(-i\omega)/\partial\omega|^2} \tag{17}$$

where $\partial\tilde{M}_{11}/\partial\omega$ is evaluated at $\omega = \omega^*$.

We now illustrate the above by application to our Lorenz ensemble, case (a) (see Fig. viii(a)). Related results for cases (b) and (c) will be reported elsewhere [11]. The black and grey solid lines in Fig. viii show $\mathrm{Re}[\tilde{M}_{11}(-i\omega)]$ and $\mathrm{Im}[\tilde{M}_{11}(-i\omega)]$ versus ω as obtained using approach (ii) with $\Delta_x = 2$ and $N = 20,000$. (We also tested other values of Δ_x up to 5, obtaining similar results, thus indicating that the perturbation is sufficiently linear.) $\mathrm{Im}[\tilde{M}_{11}(-i\omega)]$ crosses zero only at $\omega^* = 0$, where $\mathrm{Re}[\tilde{M}_{11}(-i\omega)]$ has a prominent peak. This gives a critical coupling value of -5.6 ± 0.15 in reasonable agreement with the threshold for coherence observed in Fig. viii(a). Figure viii shows the instability growth rate from Eq. (17) versus δk as a solid line, along with values observed from simulations of the full nonlinear system plotted as dots. To obtain the latter data, we first initialize the ensemble in the incoherent state by time evolution with the coupling k set to zero. We then turn on the coupling $k = -|k^*| + \delta k$, plot $\ln\langle\langle x^{(1)}\rangle\rangle$ versus t, and fit a straight line to the resulting graph during the exponential growth phase. As can be seen from Fig. viii, the result obtained from Eq. (17) agrees well with the data for $0 \geq \delta k \geq -0.6$.

Figure viii shows the result of a computation of $\langle\langle M_{11}(t)\rangle\rangle_*$ versus t by the use of approach (i) with $N = 20,000$. $\langle\langle M_{11}(t)\rangle\rangle_*$ behaves as expected for $t \leq 0.7$ (i.e., it decays with time), but past that time it shows apparent

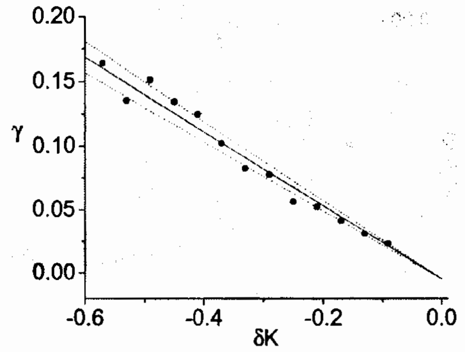

Figure 3. The growth rate γ versus δk for case (a).

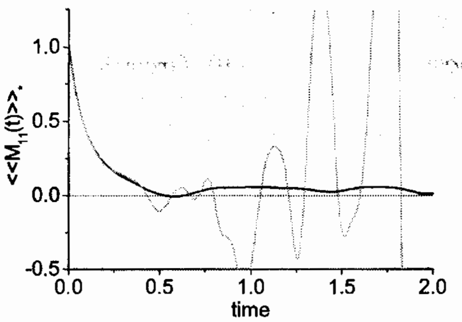

Figure 4. $\langle\langle M_{11}(t)\rangle\rangle_*$ from approach (i) versus t, $N = 20,000$.

divergent behavior. This can be understood on the basis that the individual $\mathbf{M}_i(t)$ for each orbit diverge exponentially at their largest Lyapunov exponent. By our previous arguments, however, we know that the average $\langle\langle M_{11}(t)\rangle\rangle_*$ decays. Thus the average must result in cancellation of the exponential growth components. However, since $\langle\langle M_{11}(t)\rangle\rangle_*$ decays exponentially, and the individual $\mathbf{M}_i(t)$ grow exponentially, this cancellation becomes more and more delicate as time increases. Thus, for any finite N, divergence of the method will always occur at large time. The question is whether a believable result can be obtained for a time duration that is long enough to be useful. Calculating $\mathrm{Re}[\tilde{M}_{11}(-i\omega)]$ from the result in Fig. viii by doing the Laplace transform only over the reliable range $0 \le t \le 0.7$, we obtain the result shown in Fig. viii. While there is reasonable agreement with the result from approach (ii) for $\omega \ge 0.1$, approach (i) fails to capture the important sharp increase to the peak at $\omega = 0$ which occurs for for $\omega \le 0.1$. The reason for this is that this feature would correspond to a time scale $1/\omega \sim 10$ which is well past the finite N-induced divergence in Fig. viii. Thus approach (i) yields a value of $|k^*|$ that is too large (by a factor or order 2). While approach (i) fails in this case, it can be useful in other cases depending on the strength of the divergences that the system exhibits, and particularly in periodic ensembles (e.g., case (b)) where $\mathbf{M}_i$ does not grow exponentally.

In conclusion, we have presented a general formulation for the determination of the stability of the incoherent state of a globally coupled system of continuous time dynamical units. The formalism is valid for both chaotic and periodic dynamics of the individual units. We discuss the analytic properties of $\tilde{\mathbf{M}}(s)$ and its numerical determination. We find that these are connected: analytic continuation of $\tilde{\mathbf{M}}(s)$ to the $\text{Im}(s)$ axis is necessary for application of the analysis, but in the chaotic case, can lead to numerical difficulties in determining $\tilde{\mathbf{M}}(s)$ (Fig. viii). Our numerical examples illustrate the validity of the approach, as well as practical limitation to numerical application.

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13. Other methods for calculating $\tilde{\mathbf{M}}(s)$ are also possible (see [11]). In particular, in the case of a chaotic ensemble, a technique based on unstable periodic orbits embedded

in the chaotic attractor in conjunction with cycle expansions appears to be attractive. This approach is presently under investigation.