

# THE ONSET OF SYNCHRONISM IN GLOBALLY COUPLED ENSEMBLES OF CHAOTIC AND PERIODIC DYNAMICAL UNITS

Edward Ott

*Institute for Research in Electronics and Applied Physics, Department of Physics, and  
Department of Electrical and Computer Engineering, University of Maryland, College  
Park, Maryland, 20742*

eo4@umail.umd.edu

Paul So

*Department of Physics and Astronomy and the Krasnow Institute for Advanced Study,  
George Mason University, Fairfax, Virginia, 22030*

paso@gmu.edu

Ernest Barreto

*Department of Physics and Astronomy and the Krasnow Institute for Advanced Study,  
George Mason University, Fairfax, Virginia, 22030*

ebarreto@gmu.edu

Thomas Antonsen

*Institute for Research in Electronics and Applied Physics, Department of Physics, and  
Department of Electrical and Computer Engineering, University of Maryland, College  
Park, Maryland, 20742*

antonsen@glue.umd.edu

**Abstract** A general stability analysis is given to describe the transition from incoherent to coherent behavior in globally-coupled systems of units whose individual uncoupled dynamics are chaotic and/or periodic.

**Keywords:** Synchronization, chaos, Kuramoto

## Introduction

Systems that consist of many coupled heterogeneous dynamical units are of great interest in a wide variety of situations. Past work has concentrated on the globally coupled case where the dynamics of the uncoupled units is periodic with a spread in the oscillator frequencies [1]. In that case, for sufficiently low coupling, the individual units oscillate incoherently, and, as the coupling is increased through a critical value, there is a transition to coherent dynamics in which a group of oscillators becomes locked in frequency and phase. Applications include synchrony in chirping crickets [2], flashing fireflies [3], Josephson junction arrays [4], semiconductor laser arrays [5], and cardiac pacemaker cells [6]. Recently, the case in which the individual units are chaotic has been addressed [7, 8, 9]. Here, we discuss a formalism that is capable of treating the onset of synchronism in a general system of globally coupled, heterogeneous, continuous-time dynamical units. No *a priori* assumption regarding the uncoupled dynamics of the individual units is made. Thus, one can consider chaotic or periodic dynamics of the uncoupled units, including the case where both types of units are present in the same system.

### 1. Numerical example

We first discuss a numerical example. We consider an ensemble of coupled Lorenz equations:

$$\begin{aligned} dx_i^{(1)}/dt &= \sigma(x_i^{(2)} - x_i^{(1)}) + (k/N) \sum_{i=1}^N x_i^{(1)}(t) \\ dx_i^{(2)}/dt &= r_i x_i^{(1)} - x_i^{(2)} - x_i^{(1)} x_i^{(3)} \\ dx_i^{(3)}/dt &= -bx_i^{(3)} + x_i^{(1)} x_i^{(2)}. \end{aligned} \quad (1)$$

Here,  $N \gg 1$  is the number of units in the ensemble. In our analysis, we will take  $N \rightarrow \infty$ . We set  $\sigma = 10$  and  $b = 8/3$ , and draw the parameters  $r_i$  from a uniform distribution in the interval  $[28, 52]$ , for which the uncoupled Lorenz equations yield chaotic solutions with no discernable windows. Numerical results for other ensembles, including a periodic ensemble containing a pitchfork bifurcation and a mixed ensemble with both chaotic and periodic units, are reported elsewhere [10]. Figure 1 shows an order parameter  $\bar{x}_T$  versus the coupling coefficient  $k$ , where

$$\bar{x}_T = \left\{ \frac{1}{T} \int_t^{t+T} \left( \frac{1}{N} \sum_{i=1}^N x_i^{(1)}(t') \right)^2 dt' \right\}^{1/2}$$

characterizes the degree of coherent motion. The time  $t$  is chosen large

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enough that the average is essentially independent of  $t$ . Incoherent motion of the uncoupled units as  $k$  decreases to sub- and supercritical for the onset of instability. Exponential

## 2. Generalization

### 2.1 Formalism

We now present the formalism. The equations are given in the form

where  $\mathbf{x}_i = (x_i^{(1)}, x_i^{(2)}, x_i^{(3)})^T$  and  $\mathbf{K}$  is a constant matrix.

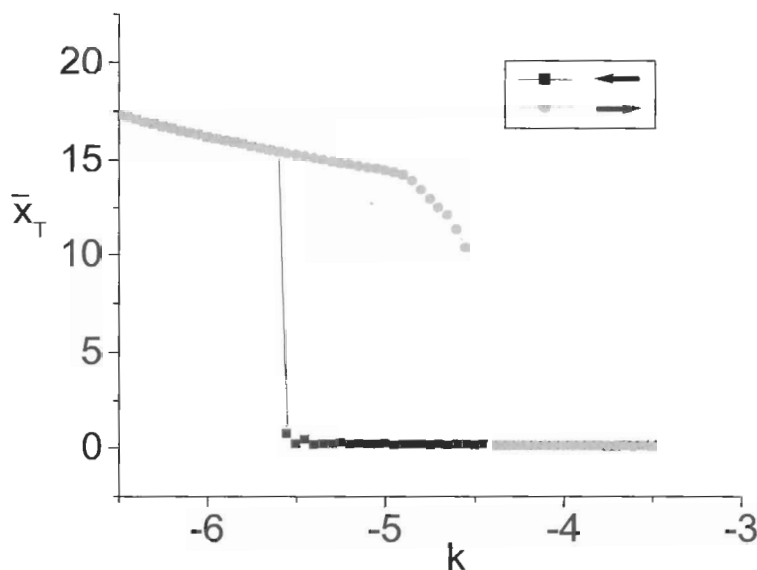


Figure 1.  $\bar{x}_T$  versus  $k$ .

enough that the system settles into its time-asymptotic dynamics, while the averaging duration  $T$  is chosen large enough that  $\bar{x}_T$  becomes essentially independent of  $T$ . Note that, if the individual units behave incoherently, the sum is close to zero by the  $x^{(1)}$  reflection symmetry of the uncoupled Lorenz equations. We observe a subcritical bifurcation as  $k$  decreases through  $k_a = -5.6$ . Other ensembles investigated include sub- and supercritical Hopf bifurcations. It is our goal to obtain a theory for the critical  $k$  values at the onset of coherence. These mark the onset of instability of the incoherent state. We are also interested in the exponential growth rates of these instabilities.

## 2. General Treatment

### 2.1 Formulation

We now present our analysis, treating the simplest case. Generalizations are given elsewhere [10]. We consider dynamical systems of the form

$$dx_i(t)/dt = \mathbf{G}(\mathbf{x}_i(t), \Omega_i) + \mathbf{K}(\langle\langle \mathbf{x} \rangle\rangle_* - \langle\langle \mathbf{x}(t) \rangle\rangle), \tag{2}$$

where  $\mathbf{x}_i = (x_i^{(1)}, x_i^{(2)}, \dots, x_i^{(q)})^T$ ;  $\mathbf{G}$  is a  $q$ -dimensional vector function;  $\mathbf{K}$  is a constant  $q \times q$  coupling matrix;  $i = 1, 2, \dots, N$ ;  $\langle\langle \mathbf{x}(t) \rangle\rangle$  is the

instantaneous average,

$$\langle\langle \mathbf{x}(t) \rangle\rangle = \lim_{N \rightarrow \infty} N^{-1} \sum_i \langle \mathbf{x}_i(t) \rangle, \quad (3)$$

and, for each  $i$ ,  $\langle \mathbf{x}_i \rangle$  is the average of  $\mathbf{x}_i$  over an infinite number of initial conditions  $\mathbf{x}_i(0)$  distributed on the attractor of the  $i$ th uncoupled system,

$$d\mathbf{x}_i/dt = \mathbf{G}(\mathbf{x}_i, \boldsymbol{\Omega}_i). \quad (4)$$

$\boldsymbol{\Omega}_i$  is a parameter vector specifying the uncoupled ( $\mathbf{K} = 0$ ) dynamics, and  $\langle\langle \mathbf{x} \rangle\rangle_*$  is the *natural measure* [11] and  $i$  average of the state of the uncoupled system. That is, to compute  $\langle\langle \mathbf{x} \rangle\rangle_*$ , we set  $\mathbf{K} = 0$ , compute the solutions to Eq. (4), and obtain  $\langle\langle \mathbf{x} \rangle\rangle_*$  from

$$\langle\langle \mathbf{x} \rangle\rangle_* = \lim_{N \rightarrow \infty} N^{-1} \sum_i \left( \lim_{\tau_0 \rightarrow \infty} \tau_0^{-1} \int_0^{\tau_0} \mathbf{x}_i(t) dt \right). \quad (5)$$

In what follows we assume that the  $\boldsymbol{\Omega}_i$  are randomly chosen from a smooth probability density function  $\rho(\boldsymbol{\Omega})$ . By construction,  $\langle\langle \mathbf{x} \rangle\rangle = \langle\langle \mathbf{x} \rangle\rangle_*$  is a solution of the  $N \rightarrow \infty$  globally coupled system (2). We call this solution the “incoherent state” because the coupling term cancels and the individual oscillators do not affect each other. We address the **stability of the incoherent state**. We envision that, as a system parameter such as the coupling strength varies, the onset of instability of the incoherent state signals the start of coherent, synchronous behavior of the ensemble.

## 2.2 Stability analysis

To perform the stability analysis, we assume that the system is in the incoherent state, so that for each  $i$  at any fixed time  $t$ ,  $\mathbf{x}_i(t)$  is distributed according to the natural measure. We then perturb the orbits  $\mathbf{x}_i(t) \rightarrow \mathbf{x}_i(t) + \delta\mathbf{x}_i(t)$ , where  $\delta\mathbf{x}_i(t)$  is an infinitesimal perturbation:

$$d\delta\mathbf{x}_i/dt = \mathbf{D}\mathbf{G}(\mathbf{x}_i(t), \boldsymbol{\Omega}_i)\delta\mathbf{x}_i - \mathbf{K}\langle\langle \delta\mathbf{x} \rangle\rangle, \quad (6)$$

where

$$\mathbf{D}\mathbf{G}(\mathbf{x}_i(t), \boldsymbol{\Omega}_i)\delta\mathbf{x}_i = \delta\mathbf{x}_i \cdot \frac{\partial}{\partial \mathbf{x}_i} \mathbf{G}(\mathbf{x}_i(t), \boldsymbol{\Omega}_i).$$

Introducing the fundamental matrix  $\mathbf{M}_i(t)$  for Eq. (6),

$$d\mathbf{M}_i/dt = \mathbf{D}\mathbf{G} \cdot \mathbf{M}_i, \quad (7)$$

where  $\mathbf{M}_i(0) \equiv \mathbf{I}$ , we can write the solution of Eq. (6) as

$$\delta\mathbf{x}_i(t) = - \int_{-\infty}^t \mathbf{M}_i(t)\mathbf{M}_i^{-1}(\tau)\mathbf{K}\langle\langle \delta\mathbf{x} \rangle\rangle_\tau d\tau, \quad (8)$$

where  $\langle\langle \delta\mathbf{x} \rangle\rangle_\tau$  is through Eq. (7) uncoupled nonlinear conditions  $\mathbf{x}_i(0)$

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where  $\langle\langle \delta \mathbf{x} \rangle\rangle_\tau$  signifies that  $\langle\langle \delta \mathbf{x} \rangle\rangle$  is evaluated at time  $\tau$ . Note that, through Eq. (7),  $\mathbf{M}_i$  depends on the unperturbed orbits  $\mathbf{x}_i(t)$  of the uncoupled nonlinear system (4), which are determined by their initial conditions  $\mathbf{x}_i(0)$  (distributed according to the natural measure).

Assuming that  $\langle\langle \delta \mathbf{x} \rangle\rangle$  evolves exponentially in time (i.e.,  $\langle\langle \delta \mathbf{x} \rangle\rangle = \Delta e^{st}$ ), Eq. (8) yields

$$\{\mathbf{I} + \tilde{\mathbf{M}}(s)\mathbf{K}\}\Delta = 0, \tag{9}$$

where  $s$  is complex, and

$$\tilde{\mathbf{M}}(s) = \left\langle \left\langle \int_{-\infty}^t e^{-s(t-\tau)} \mathbf{M}_i(t) \mathbf{M}_i^{-1}(\tau) d\tau \right\rangle \right\rangle_*. \tag{10}$$

Thus the dispersion function determining  $s$  is

$$D(s) = \det\{\mathbf{I} + \tilde{\mathbf{M}}(s)\mathbf{K}\} = 0. \tag{11}$$

In order for Eqs. (9) and (11) to make sense, the right side of Eq. (10) must be independent of time. We now demonstrate this, and express  $\tilde{\mathbf{M}}(s)$  in a more convenient form. Writing the dependence of  $\mathbf{M}_i$  in Eq. (10) on the initial condition explicitly, we have from the definition of  $\mathbf{M}_i$ ,

$$\mathbf{M}_i(t, \mathbf{x}_i(0)) \mathbf{M}_i^{-1}(\tau, \mathbf{x}_i(0)) = \mathbf{M}_i(t - \tau, \mathbf{x}_i(\tau)) = \mathbf{M}_i(T, \mathbf{x}_i(t - T)),$$

where  $T = t - \tau$ . Using this in Eq. (10) we obtain

$$\tilde{\mathbf{M}}(s) = \left\langle \left\langle \int_0^\infty e^{-sT} \mathbf{M}_i(T, \mathbf{x}_i(t - T)) dT \right\rangle \right\rangle_*.$$

Our solution requires that this integral converge. Since the growth of  $\mathbf{M}_i$  with increasing  $T$  is dominated by  $h_i$ , the largest Lyapunov exponent for orbit  $\mathbf{x}_i$ , we require  $\text{Re}(s) > \Gamma$ , where  $\Gamma = \max_{\mathbf{x}_i, \Omega_i}(h_i)$ . In this case, the integral converges exponentially and uniformly, and the order of the integration and the averaging operations can be interchanged:

$$\tilde{\mathbf{M}}(s) = \int_0^\infty e^{-sT} \langle\langle \mathbf{M}_i(T, \mathbf{x}_i(t - T)) \rangle\rangle_* dT. \tag{12}$$

Here the only dependence on  $t$  is through the initial condition  $\mathbf{x}_i(t - T)$ . The average over the initial conditions, which are distributed according to the invariant natural measure, ensures that  $\langle\langle \mathbf{M}_i(T, \mathbf{x}_i(t - T)) \rangle\rangle_*$  is independent of  $t$ . Thus  $\tilde{\mathbf{M}}$  is the Laplace transform of  $\langle\langle \mathbf{M} \rangle\rangle_*$ . As shown below,  $\tilde{\mathbf{M}}(s)$  can be analytically continued into  $\text{Re}(s) < \Gamma$ .

### 2.3 Discussion

$\tilde{\mathbf{M}}(s)$  depends only on the solution of the linearized *uncoupled* system (Eq. (7)). Hence the utility of the dispersion function  $D(s)$  given by Eq. (11) is that it determines the linearized dynamics of the globally coupled system in terms of those of the individual uncoupled systems.

Consider the  $k$ th column of  $\langle\langle\mathbf{M}(t)\rangle\rangle_*$ , denoted  $[\langle\langle\mathbf{M}(t)\rangle\rangle_*]_k$ , which we interpret as follows. Assume that for each of the uncoupled systems  $i$  in Eq. (4), we have a cloud of an infinite number of initial conditions distributed according to the natural measure on the uncoupled attractor. Then, at  $t = 0$ , we apply an equal infinitesimal displacement  $\delta_k$  in the direction  $k$  to each orbit in the cloud. The quantity  $[\langle\langle\mathbf{M}\rangle\rangle_*]_k \delta_k$  gives the time evolution of the  $i$ -averaged perturbation of the centroid of the clouds as the perturbed orbits evolve back to the attractor and redistribute themselves on the attractor.

We now argue that  $\langle\langle\mathbf{M}\rangle\rangle_*$  decays to zero exponentially with increasing time. We consider the general case where the support of the smooth density  $\rho(\Omega)$  contains open regions of  $\Omega$  for which Eq. (4) has attracting periodic orbits as well as a positive measure of  $\Omega$  on which Eq. (4) has chaotic orbits. Numerical experiments on chaotic attractors (including structurally unstable attractors) generally show that they are strongly mixing; i.e., a cloud of many particles rapidly arranges itself on the attractor according to the natural measure. Thus, for each  $\Omega_i$  giving a chaotic attractor, it is reasonable to assume that the average of  $\mathbf{M}_i$  over initial conditions  $\mathbf{x}_i(0)$ , denoted  $\langle\mathbf{M}_i\rangle_*$ , decays exponentially. For a periodic attractor, however,  $\langle\mathbf{M}_i\rangle_*$  does not decay, but periodic orbits exist in open regions of  $\Omega$ , and when averaged over  $\Omega$ , there is the possibility that with increasing time, cancellation, causing decay, occurs via the process of "phase mixing". For this case we appeal to an example. The explicit computation of  $\langle\mathbf{M}_i\rangle_*$  for a simple model limit cycle ensemble is given in Ref. [10]. The result is

$$\langle\mathbf{M}_i\rangle_* = \frac{1}{2} \begin{bmatrix} \cos \Omega_i t & -\sin \Omega_i t \\ \sin \Omega_i t & \cos \Omega_i t \end{bmatrix},$$

and indeed this oscillates and does not decay to zero. However, if we average over the oscillator distribution  $\rho(\Omega)$  we obtain

$$\langle\langle\tilde{\mathbf{M}}\rangle\rangle_* = \frac{1}{2} \begin{bmatrix} c(t) & -s(t) \\ s(t) & c(t) \end{bmatrix},$$

where  $c(t) = \int \rho(\Omega) \cos \Omega t d\Omega$  and  $s(t) = \int \rho(\Omega) \sin \Omega t d\Omega$ . For any analytic  $\rho(\Omega)$  these integrals decay exponentially with time. Thus, for sufficiently smooth  $\rho(\Omega)$ , there is reason to believe that  $\langle\langle\mathbf{M}\rangle\rangle_*$  decays

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to zero with increasing time. Conjecturing this decay to be exponential,  $\|\langle\langle\mathbf{M}(t)\rangle\rangle_*\| < \kappa e^{-\xi t}$  for positive constants  $\kappa$  and  $\xi$ , we see that the integral in Eq. (12) converges for  $\text{Re}(s) > -\xi$ . This conjecture is supported by our numerical results. Thus, using analytic continuation, we can regard Eq. (12) as valid for  $\text{Re}(s) > -\xi$ .

### 3. Numerical implementation

In order to apply Eq. (11) to a given situation, it is necessary to numerically approximate the matrix  $\tilde{\mathbf{M}}(s)$ . We consider two possible candidate approaches.

Approach (i): First approximate the natural measure on each attractor  $i$  by a large finite number of orbits initially distributed according to the natural measure. For each initial condition, obtain  $\mathbf{x}_i(t)$  from Eq. (4). Use these solutions in  $\mathbf{DG}$  and solve Eq. (7). Then average over the natural measure and  $i$  to obtain  $\langle\langle\mathbf{M}(t)\rangle\rangle_*$ , and apply the Laplace transform as in Eq. (12).

Approach (ii): Since  $\langle\langle\mathbf{M}\rangle\rangle_*$  is the response to an impulse (i.e., the sudden displacement of each orbit), its Laplace transform  $\tilde{\mathbf{M}}(s)$  multiplied by  $\exp(st)$  is the response to the drive  $\mathbf{I}\exp(st)$  added to the right side of Eq. (6). This suggests the following numerical procedure for finding  $\tilde{\mathbf{M}}(s)$ . Solve

$$\frac{d\tilde{\mathbf{x}}_i^{(c,s)}}{dt} = \mathbf{G}(\tilde{\mathbf{x}}_i^{(c,s)}, \boldsymbol{\Omega}_i) + \Delta_k \mathbf{a}_k e^{\sigma t} \begin{cases} \cos \omega t \\ \sin \omega t, \end{cases} \quad (13)$$

where  $s = \sigma - i\omega$  and  $\mathbf{a}_k$  is a unit vector in the direction  $k$ . For large  $t$ , but  $\Delta_k \exp(\sigma t)$  still small in the time interval  $(0, t)$ , we can regard the average response as approximately linear. Thus, the  $k$ th column of  $\tilde{\mathbf{M}}(s)$  is

$$[\tilde{\mathbf{M}}(s)]_k \simeq \Delta_k^{-1} e^{-st} [\langle\langle\mathbf{x}\rangle\rangle_* - \langle\langle\tilde{\mathbf{x}}\rangle\rangle], \quad (14)$$

where  $\tilde{\mathbf{x}}_i = \tilde{\mathbf{x}}_i^{(c)} - i\tilde{\mathbf{x}}_i^{(s)}$ . Numerically,  $\langle\langle\tilde{\mathbf{x}}\rangle\rangle$  can be approximated using a large finite number of orbits. In Ref. [8], a technique equivalent to this with  $s$  taken to be imaginary ( $s = -i\omega$ ) was used to obtain marginal stability.

### 4. Example

In the example introduced in Section 1, we chose the coupling such that only the  $(1, 1)$  element of  $\tilde{\mathbf{M}}(s)$  was nonzero. Thus Eq. (11) reduces to

$$1 + \tilde{M}_{11}(-i\omega)k = 0 \quad (15)$$

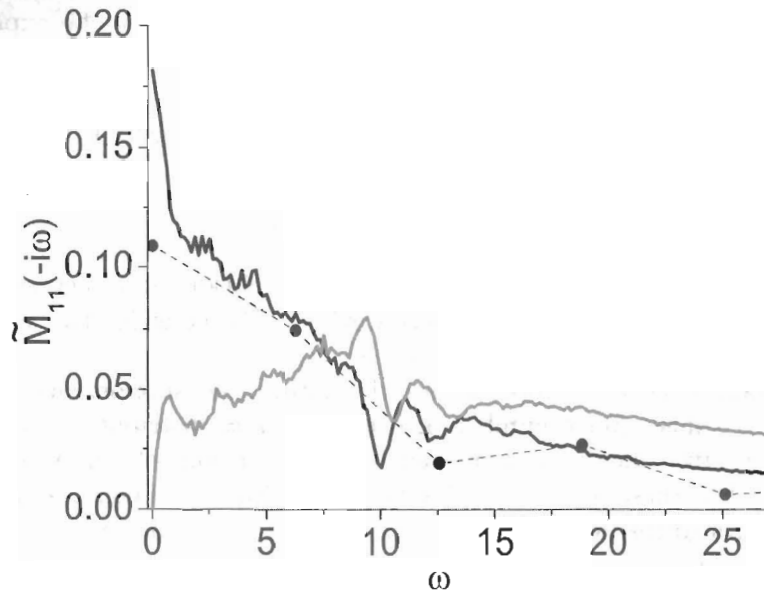


Figure 2.  $\tilde{M}_{11}(-i\omega)$  versus  $\omega$ . The solid black line is  $\text{Re}[\tilde{M}_{11}]$ , approach (ii); the solid grey line is  $\text{Im}[\tilde{M}_{11}]$ , approach (ii); and the dashed line is  $\text{Re}[\tilde{M}_{11}]$ , approach (i).

where we have set  $s = -i\omega$ . Solving  $\text{Im}\tilde{M}_{11}(-i\omega) = 0$  yields possibly multiple roots  $\omega = \omega^*$ , which, when reinserted into Eq. (15), yield possible values  $k = k^* = -[\tilde{M}_{11}(-i\omega^*)]^{-1}$  for the critical coupling strengths. To determine which of these are relevant, we envision that as  $k$  is increased or decreased from zero, a critical coupling value is encountered at which the incoherent state first becomes unstable. Hence we are interested in the roots  $\omega_{a,b}^*$  corresponding to the smallest  $|k^*|$  for  $k^*$  both negative ( $k^* = -|k_a^*|$ ) and positive ( $k^* = k_b^* > 0$ ). Growth rates and frequency shifts from  $\omega^*$  can also be simply obtained for  $k$  near  $k^*$  by setting  $k = k^* + \delta k$ ,  $s = -i(\omega^* + \delta\omega) + \gamma$  and expanding Eq. (11) for small  $\delta k$ ,  $\delta\omega$ , and  $\gamma$ ; e.g.,

$$\gamma = -\frac{\delta k}{(k^*)^2} \frac{\partial \text{Im}[\tilde{M}_{11}(-i\omega)]/\partial\omega}{|\partial\tilde{M}_{11}(-i\omega)/\partial\omega|^2} \quad (16)$$

where  $\partial\tilde{M}_{11}/\partial\omega$  is evaluated at  $\omega = \omega^*$ .

The black and grey solid lines in Fig. 2 show  $\text{Re}[\tilde{M}_{11}(-i\omega)]$  and  $\text{Im}[\tilde{M}_{11}(-i\omega)]$  versus  $\omega$  for the chaotic Lorenz ensemble in Eq. (1). Here, we used approach (ii) with  $\Delta_x = 2$  and  $N = 20,000$ .  $\text{Im}[\tilde{M}_{11}(-i\omega)]$  crosses zero only at  $\omega^* = 0$ , where  $\text{Re}[\tilde{M}_{11}(-i\omega)]$  has a prominent peak.

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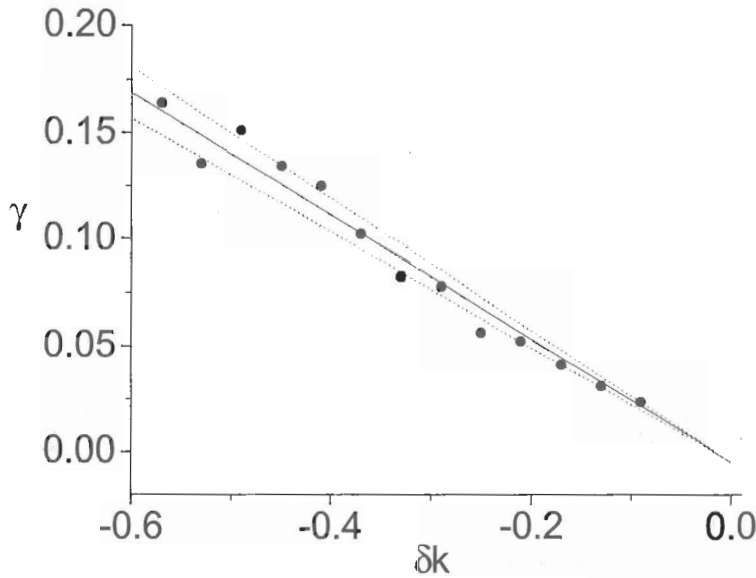


Figure 3. The growth rate  $\gamma$  versus  $\delta k$ .

This gives a critical coupling value of  $-5.6 \pm 0.15$  in reasonable agreement with the threshold for coherence observed in Fig. 1. Figure 3 shows the instability growth rate from Eq. (16) versus  $\delta k$  as a solid line, along with values observed from simulations of the full nonlinear system plotted as dots. To obtain the latter data, we first initialize the ensemble in the incoherent state by time evolution with the coupling  $k$  set to zero. We then set  $k = -|k^*| + \delta k$ , plot  $\ln \langle x^{(1)} \rangle$  versus  $t$ , and fit a line to the resulting graph during the exponential growth phase. These data agree well with Eq. (16) for  $0 \geq \delta k \geq -0.6$  (Fig. 3).

Figure 4 shows  $\langle \langle M_{11}(t) \rangle \rangle_*$  versus  $t$  obtained by approach (i).  $\langle \langle M_{11}(t) \rangle \rangle_*$  decays as expected for  $t \leq 0.7$ , but then shows apparent divergent behavior. This can be understood on the basis that the individual  $M_i(t)$  for each orbit diverge exponentially at their largest Lyapunov exponent. Since  $\langle \langle M_{11}(t) \rangle \rangle_*$  decays, the averaging process must result in a cancellation of the exponential growth components, and this cancellation becomes more and more delicate as time increases. Thus, for any finite  $N$ , divergence of the method will always occur at large time. Calculating  $\text{Re} \tilde{M}_{11}(-i\omega)$  from the result in Fig. 4 by doing the Laplace transform only over the reliable range  $0 \leq t \leq 0.7$ , we obtain the result shown in Fig. 2 as the dashed curve. While there is reasonable agreement with the result from approach (ii) for  $\omega \geq 0.1$  (in Fig. 2), approach (i) fails

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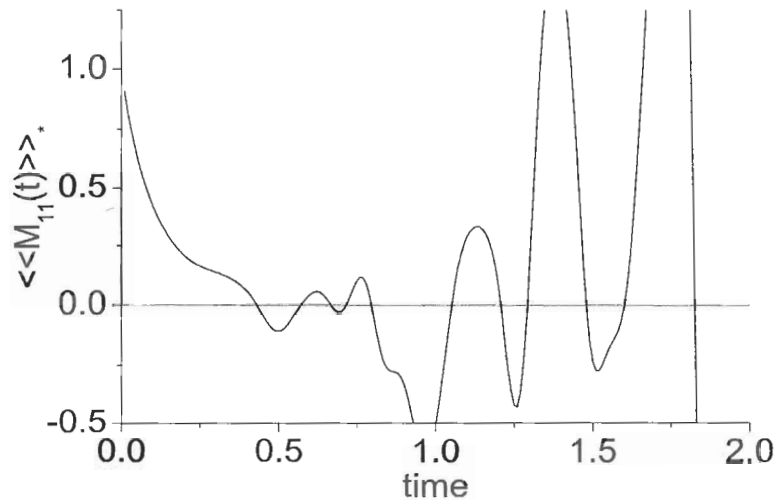


Figure 4.  $\langle\langle M_{11}(t) \rangle\rangle^*$  from approach (i) versus  $t$ ,  $N = 20,000$ .

to capture the important sharp increase to the peak at  $\omega = 0$  which occurs for  $\omega \leq 0.1$ . This feature corresponds to a time scale  $1/\omega \sim 10$  which is well past the finite  $N$ -induced divergence in Fig. 4. Thus approach (i) yields a value of  $|k^*|$  that is too large (by a factor of order 2). While approach (i) fails in this case, it can be useful in other cases depending on the strength of the divergences that the system exhibits, and particularly in the case of periodic ensembles where  $\mathbf{M}_i$  does not grow exponentially [10].

## 5. Conclusion

We have presented a general formulation for the determination of the stability of the incoherent state of a globally coupled system of continuous time dynamical units. The formalism is valid for both chaotic and periodic dynamics of the individual units. We discuss the analytic properties of  $\mathbf{M}(s)$  and its numerical determination. We find that these are connected: analytic continuation of  $\tilde{\mathbf{M}}(s)$  to the imaginary axis is necessary for application of the analysis, but in the chaotic case, can lead to numerical difficulties in determining  $\tilde{\mathbf{M}}(s)$  (Fig. 4). Our numerical example illustrates the validity of the approach, as well as practical limitation to numerical application.

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## References

- [1] Y. Kuramoto, *Chemical Physics* (Springer-Verlag, Berlin, 1975); (1984); A. T. (1980); for a review see *Phys. Rev. Lett.* **D 143**, 1 (2000).
- [2] T. J. Walker, *Phys. Rev. Lett.* **58**, 1000 (1987).
- [3] J. Buck, Q. R. (1984).
- [4] K. Wiesenfeld, *Phys. Rev. Lett.* **69**, 1256 (1992).
- [5] D. V. Ramanao, *Phys. Rev. Lett.* **81**, 4640 (1998).
- [6] C. S. Peskin, *Mathematical Biophysics* (Springer, 2000).
- [7] A. S. Pikovsky, *Phys. Rev. Lett.* **77**, 1224 (1996).
- [8] H. Sakaguchi, *Phys. Rev. Lett.* **77**, 1224 (1996).
- [9] D. Topaj, W. (1998).
- [10] E. Ott, P. So. (1998).
- [11] E. Ott, *Chaos* **8**, 10 (1998).

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## References

- [1] Y. Kuramoto, in *International Symposium on Mathematical Problems in Theoretical Physics*, edited by H. Araki, Lecture Notes in Physics, Vol. 39 (Springer, Berlin, 1975); *Chemical Oscillators, Waves and Turbulence* (Springer, Berlin, 1984); A. T. Winfree, *The Geometry of Biological Time* (Springer, New York, 1980); for a review of work on the Kuramoto model, see S. H. Strogatz, *Physica D* **143**, 1 (2000).
- [2] T. J. Walker, *Science* **166**, 891 (1969).
- [3] J. Buck, *Q. Rev. Biol.* **63**, 265 (1988).
- [4] K. Wiesenfeld, P. Colet and S. H. Strogatz, *Phys. Rev. Lett.* **76**, 404 (1996).
- [5] D. V. Ramana Reddy, A. Sen and G. L. Johnston, *Phys. Rev. Lett.* **80**, 5109 (1998).
- [6] C. S. Peskin, *Mathematical Aspects of Heart Physiology* (Courant Institute of Mathematical Sciences, New York, 1975).
- [7] A. S. Pikovsky, M. G. Rosenblum and J. Kurths, *Europhys. Lett.* **34**, 165 (1996).
- [8] H. Sakaguchi, *Phys. Rev. E* **61**, 7212 (2000).
- [9] D. Topaj, W. -H. Kye and A. Pikovsky, *Phys. Rev. Lett.* **87**, 074101 (2001).
- [10] E. Ott, P. So, E. Barreto, and T.M. Antonsen, *Physica D* **173**, 226-258 (2002).
- [11] E. Ott, *Chaos in Dynamical Systems* (Cambridge Univ. Press, 1993), Chapter 3.