CONTROL OF CHAOS:
IMPACT OSCILLATORS AND TARGETING

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Abstract. We present two applications of chaos control techniques that can be of importance in mechanical systems. First, we apply chaos control to select a desired sequence of impacts in a map that captures the universal properties of impact oscillators near grazing. Next we describe a targeting method that can significantly reduce the chaotic transients that precede stabilization when these control methods are used.

1. Introduction

Recently, the application of chaos control techniques to physical systems has commanded increasing attention. In this work we describe the application of these methods to the impact oscillator, a mechanical system of great

importance. We also describe a targeting method that can improve the utility of control techniques when applied to higher dimensional chaotic systems.

As impact oscillators is a forced vibrating mechanical system which undergoes a sequence of contacts with motion-limiting constraints. The dynamics is therefore smooth motion, governed by a differentiable equation, interrupted by a series of non-smooth collisions. The collisions introduce nonlinearity into the system. Impact oscillators are used to model a variety of different systems arising in engineering (for example, moored ships colliding with fenders, forced mechanical systems with clearances such as rattling gears, and railway vehicles [1, 2]).

Mathematically, impact oscillators constitute a subclass of dynamical systems that do not satisfy the usual smoothness assumptions. These discontinuities are responsible for new forms of behavior not found in smooth dynamical systems, particularly in the limit of low velocity or grazing impacts [1-8].

In engineering, systems are modelled and investigated in order to identify and avoid unacceptable responses. For impact systems, it is necessary to avoid high velocity impacts as these cause the greatest wear or damage to components. This can be accomplished by the well-known techniques of chaos control [9]. The flexibility provided by chaos allows us to select particular trajectories with a desirable sequence of impacts. This can be advantageous in many technological applications of impact oscillators.

In this work we apply the method of Ott, Grebogi and Yorke to control chaotic impacts in the Nordmark map [3, 10, 11] \( \dot{x}_{n+1}, y_{n+1} = F_D(x_n, y_n) \), where

\[
F_D(x, y) = \begin{cases} 
(ox + y + r, -\gamma x) & \text{for } x \leq 0 \\
(-\sqrt{x} + y + r, -\gamma^2 x) & \text{for } x > 0.
\end{cases} \tag{1}
\]

This is a piecewise continuously differentiable map that models the behavior of a sinusoidally forced linear oscillator experiencing impacts at a hard wall. It is obtained by expanding solutions of the system in the neighborhood of a grazing orbit [3], i.e., of an orbit that just touches the wall with zero velocity. The map captures the universal properties of the dynamics in the regime of low velocity impacts. The equivalence with the physical system is as follows: \( x_n \) and \( y_n \) are transformed coordinates in the position-velocity space \( (\xi, \dot{\xi}) \) evaluated at times \( t_n = 2n\pi/\omega \), where \( \omega \) is the frequency of the external forcing. The quantity \( r^2 \) is the restitution coefficient of the impacts, and \( \rho \) is related to \( b_0 \), the amplitude of the external force. The parameters \( \alpha \) and \( \gamma \) depend on the intrinsic properties of the oscillator such that the limit \( \gamma \to 0 \) corresponds to a large coefficient of friction, and \( \gamma r^2 = 1 \) gives the opposite limit of zero dissipation. For physical systems (with positive friction) we have \( 0 < \gamma \).

The top expression in (1) no impact between times: expression applies. Thus, \( t_n \) a square root nonlinearity.

2. Control of the Impact

The control technique enables one to select a pre chaotic attractor by a set of accessible parameter [9]. First one chooses a chaotic attractor according one defines a small region starting with almost any region by ergodicity. When able control parameters is of the desired unstable or stabilization of different \( \mu \) of the parameter. This is embedded within it a large We choose a single control the driving. The grazing is neighborhood of grazing this parameter is increased.

By applying the OGY a periodic trajectories with injection of impacts per period; expressions for the posterior information needed for apriori data [9, 13]. Here, it orbits [10], i.e., periodic per period.

For systems with parameters of stable maximal periodic positive values [10, 11]. It from the succeeding we have an infinite cascade of such period, accumulating on.
method that can improve the higher dimensional chaotic mechanical systems which un- }n{iting constraints. The dynamics (with positive friction) we have [10-12] 
\[ 0 < \gamma < 1, \quad -2\sqrt{\gamma} < \alpha < 1 + \gamma. \] (2)

The top expression in (1), valid for \( x \leq 0 \), governs the system if there is no impact between time \( t_n \) and \( t_{n+1} \). Otherwise, \( x > 0 \) and the second expression applies. Thus, the effect of impacts in the system is modelled by a square root nonlinearity.

2. Control of the Impact Oscillator

The control technique of Ott, Grebogi and Yorke has the feature that it enables one to select a predetermined time-periodic behavior embedded in a chaotic attractor by making only small time-dependent perturbations to a set of accessible parameters of the system. The basic idea is as follows [9]. First one chooses a desirable unstable periodic orbit embedded in the chaotic attractor according to some set of performance criteria. Second, one defines a small region around the desired periodic orbit. A trajectory starting with almost any initial condition eventually falls into this small region by ergodicity. When this occurs, one applies perturbations to available control parameters so as to move the orbit onto the stable manifold of the desired unstable orbit. The flexibility of the method allows for the stabilization of different periodic orbits for the same set of nominal values of the parameter. This is possible because a chaotic attractor typically has embedded within it a large number of different unstable periodic orbits. We choose a single control parameter, \( \rho \). This characterizes the strength of the driving. The grazing state corresponds to \( \rho = 0 \), and dynamics in the neighborhood of grazing is given by \( |p| < 1 \). Bifurcations occur as the parameter \( \rho \) is increased through \( \rho = 0 \) with \( \gamma \) and \( \alpha \) held fixed.

By applying the OGY algorithm to the Nordmark map, one can stabilize periodic trajectories with an arbitrary number and an arbitrary distribution of impacts per period. This is so even if it is not possible to get analytic expressions for the position of the physical components. Also, the necessary information needed for applying control can be extracted purely from measured data [9, 13]. Here, for simplicity, we consider only maximal periodic orbits [10], i.e., periodic trajectories for which there is exactly one impact per period.

For systems with parameters in the region \( 4\gamma + \frac{3}{4} < 1 < \frac{3\gamma + 1}{2} \), windows of stable maximal periodic orbits are encountered as \( \rho \) is decreased from positive values [10, 11]. In particular, a window of period \( p \) is separated from the succeeding window of period \( p + 1 \), by a band of chaos. There is an infinite cascade of such windows of decreasing width in \( \rho \) and increasing period, accumulating on \( \rho = 0^+ \). This is illustrated by the bifurcation
Figure 1. Bifurcation diagram for $(\gamma, \alpha) = (0.05, 0.65)$ and $r^2 = 1$ for small positive $\rho$ values.

diagram of Fig. 1, obtained for $(\gamma, \alpha) = (0.05, 0.65)$ and $r^2 = 1$ for small positive $\rho$ values. Here we can avoid the presence of chaotic impacts for $\rho > 0$ by applying control. As an example we take $\rho = \exp(-9.2)$, on the left band of chaos in Fig. 1. Here we have unstable maximal orbits up to period $M = 8$ embedded in the chaotic attractor.

Fig. 2 illustrates control of these periodic orbits. We plot the $x$-coordinate as a function of time. The parameter perturbations were programmed to successively control the seven different periodic orbits. Control for the $M = 2$ maximal orbit was turned on after 3000 free iterations. Each window was controlled for 500 iterations before switching to the next orbit. The figure shows that the time to achieve control is almost negligible in this case, with no apparent transients between switches. The maximum allowed parameter perturbation is $\delta = 10^{-4}$. Thus it is possible to convert chaotic impacts to controlled periodic orbits by applying only small perturbations $|\delta\rho| < 10^{-4}$ to the parameter $\rho$.

For parameters in the region $\frac{3}{2} \gamma + \frac{1}{2} < \alpha < 1 + \gamma$, there is an interval of $\rho$ values occupied entirely by a chaotic attractor. As $\rho$ increases from zero, this interval terminates at a stable maximal orbit of some period $M_0$ [10]. This attractor has embedded within it unstable maximal periodic orbits of increasing period as $\rho$ approaches zero. An example of a bifurcation diagram for this case is shown in Fig. 3, obtained for $(\gamma, \alpha) = (0.15, 1)$ and $r^2 = 1$. Here we can control chaos by stabilizing any of the maximal orbits which are present for positive values of $\rho$. For $\rho = 0.05$, we have unstable
and \( r^2 = 1 \) for small positive \( r \).

0.65) and \( r^2 = 1 \) for small values of chaotic impacts for \( \rho = \exp(-9.2) \), on the table maximal orbits up to \( \gamma \). We plot the \( x \)-coordinate perturbations were present periodic orbits. Control 3000 free iterations. Each switching to the next orbit, \( \delta \) is almost negligible in this thesis. The maximum allowed is possible to convert chaotic using only small perturbations 1 + \( \gamma \), there is an interval of \( \rho \). As \( \rho \) increases from zero, a bit of some period \( M_0 \), an example of a bifurcation ed for \( (\gamma, \alpha) = (0.15, 1) \) and is any of the maximal orbits \( \rho = 0.05 \), we have unstable

Figure 2. Successive control of unstable maximal periodic orbits for \( \rho = \exp(-9.2) \), starting with period \( M = 2 \). The maximum parameter perturbation is \( \delta = 10^{-3} \).

Figure 3. Bifurcation diagram for \( \gamma = 0.15, \alpha = 1 \) and \( r^2 = 1 \).
3. Targeting of Periodic Orbits

The method described above relies on the natural ergodicity of chaotic dynamics to bring a trajectory into the vicinity of a desired stable periodic orbit where it can be actively controlled. In applications involving higher dimensional systems, the times required for this to happen may be prohibitively long. For example, Romeiras et al. [9] have have applied the method to a four-dimensional map that describes a kicked double rotor [14], shown in Figure 5. They showed that control can be achieved by using only one control parameter (even when the attractor has two positive Lyapunov exponents). However, some unstable periodic orbits require several hundred thousand iterations before stabilization is achieved [15].

Targeting is a slightly different version of the control problem. We assume that we are given some initial condition on the attractor, and we wish to rapidly direct the resulting trajectory to a small region about some specified point on the attractor. Because of the inherent exponential sensitivity of chaotic time evolutions to perturbations, one expects that this can be accomplished using only small controlling adjustments of one or more available system parameters.

This was demonstrated in the case of a two-dimension laboratory experiment for a dimensional map [17]. Kost targeting procedure that can such as the double rotor can be realized.

Because the dimension of the attractor chosen in Romeiras becomes nearest neighbors is as $N^{1/2.8}$. This implies that the required to come within $10^{-5}$ control procedure described $10^{-4}$ of the target is less than...
Figure 5. The Kicked Double Rotor. A massless rod of length $l_1$ pivots about the stationary point $P_1$. A second massless rod of length $l_2$ is mounted on pivot $P_2$, which is in turn mounted at the end of the first rod. Periodic impacts kick $f(t) = \sum_{i=-\infty}^{\infty} \mu_i \delta(t-n)$ are applied at an angle $\phi$ as shown. The state of the system immediately after the ($n+1$)th kick is given by a four-dimensional map of the form $X_{n+1} = MY_n + X_n$ and $Y_{n+1} = LX_n + G(X_{n+1})$, where $X = (\theta_1, \theta_2, Y_1, Y_2)^T$ are the two angular position coordinates, $Y = (r_1, r_2)^T$ are the corresponding angular velocities, and $G(X)$ is a nonlinear function. $M$ and $L$ are both constant matrices which involve the coefficients of friction at the two pivots and the moments of inertia of the rotor. Gravity is absent. Control parameters at time $n$ are $\rho_n = 9.0 + \Delta \rho_n$ and $\omega_n = 9.0 + \Delta \omega_n$, with $|\Delta \rho_n|/\rho_n \leq 0.1$ and $|\Delta \omega_n| \leq 0.5$. We take $l_i = 1/\sqrt{2}$, and set all other parameters to 1. For further details, see Ref. [14].

available system parameters.

This was demonstrated theoretically and in numerical experiments for the case of a two-dimensional map by Shinbrot et al. [16], and also in a laboratory experiment for which the dynamics were approximated by a one-dimensional map [17]. Kostelich et al. [18] developed an extension of the targeting procedure that can be applied to higher dimensional systems, such as the double rotor map.

Because the dimension of the double rotor attractor (for the set of parameters chosen in Romeiras, et al.) is about 2.3, the average distance between nearest neighbors in a subset of $N$ points on the attractor scales as $N^{-1/2.3}$. This implies that, on average, $10^{11}$ iterations of the map are required to come within $10^{-4}$ of the target without the control. Since the control procedure described in [18] can steer the initial condition to within $10^{-4}$ of the target in less than $10^{9}$ steps, the method can achieve the target
about $10^9$ times faster than the uncontrolled chaotic process.

The method works in two steps. First, information is learned about the system by observing a very long chaotic orbit, and constructing targeting trees as follows. The map is iterated from a random initial condition while keeping in memory a short history of the iterates encountered (for example, 10 consecutive points), until the orbit lands within a suitable tolerance distance of the target. This point, together with the recorded pre-iterates, comprise the trunk path of the tree, and are stored in memory. The map is then iterated again, still keeping track of a brief iterate history, until the orbit lands near any one of the points already in the tree. When this happens, a new path is added as a branch. Continuing in this way, a tree is built with a hierarchy of branches: the trunk path is level 1; level 2 branches are those that are rooted at some point in the trunk path; level 3 branches are rooted at a level 2 branch, and so on. The objective is to build a tree with enough branches such that a typical chaotic orbit lands near a point in the tree after a small number of iterations.

Once a sufficiently large targeting tree has been built, a chaotic orbit can be steered along the tree to the target. One applies small changes to available parameters to steer the orbit to the stable manifold of a point in the tree. (The stable manifold $S$ associated with a typical point $x$ is stable in the sense that $|F^n(x) - F^n(y)| \to 0$ as $n \to \infty$ whenever $y \in S$.) When the method is successful, the dynamics of the system carry the orbit of the perturbed point close to an orbit that leads directly to the target. Additional details on the method are given in [18].

The targeting algorithm can be combined with the OGY control method to provide a means to rapidly switch a given chaotic process between pre-specified periodic orbits. That is, the targeting procedure can be used to steer a given initial condition on the attractor to a neighborhood of one of the periodic orbits, then the OGY control method can be used to stabilize the system near the periodic orbit. The combined method is discussed in [15], and the results of its application to the double rotor are shown in Figures 6 and 7.

4. Conclusions

In summary, we have shown that chaotic dynamics in impact oscillators can be converted into motion on a desired periodic orbit by using only small parameter perturbations. In higher dimensional systems, it is possible to employ a targeting technique to reduce the length of the chaotic transients that precede stabilization. These results can be of importance in technological applications.

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A chaotic process is learned about the system and constructing targeting domain initial condition while encountered (for example, within a suitable tolerance to the recorded pre-iterated, stored in memory. The map briefly iterate history, until steady in the tree. When this tuing in this way, a tree is built, a chaotic orbit is computed and the branch 3 branches objective is to build a tree orbit lands near a point been built, a chaotic orbit is computed and the branch 3 branches objective is to build a tree orbit lands near a point with typical point \( y \) is 

\[ y_n \to y \text{ whenever } y \in S \]

the system carry the orbit directly to the target. [8].

With the OGY control method, chaotic process between pre-stable manifold of a point with a typical point \( y \) is 

\[ y_n \to y \text{ whenever } y \in S \]

the system carry the orbit directly to the target. [8].

Figure 6. Graph illustrating switching between five different fixed points. The \( \delta \) coordinate of the state is plotted versus iteration. Here we rely on ergodicity to bring the orbit close to the desired UPO. The 5th fixed point required 154,485 iterations to be stabilized and is not shown.

Figure 7. Improvements of up to four orders of magnitude in the switching times is evident.

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References

HILL'S PROBLEM A

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Projects exist for expeditions using jumping robots. These are under the simultaneous action of satellite (Phobos or Diamon) and surface.

\[ \dot{\mathbf{r}} = \mathbf{F}(\mathbf{r}, t) \]

Fig 1

Its origin coincides with the main co-ordinate system, the usual face towards the planet.
The satellite is considered as the vicinity of the satellite is geometrically given.

These equations are on the surface of the satellite. These impacts an.

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