Ergod. Th. & Dynam. Sys. (2002), **22**, 953–972
DOI: 10.1017/S0143385702000615

Explosions: global bifurcations at heteroclinic tangencies

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(*Received* 5 *November* 2000 *and accepted in revised form* 21 *September* 2001)

Abstract. We investigate bifurcations in the chain recurrent set for a particular class of one-parameter families of diffeomorphisms in the plane. We give necessary and sufficient conditions for a discontinuous change in the chain recurrent set to occur at a point of heteroclinic tangency. These are also necessary and sufficient conditions for an Ω-explosion to occur at that point.

1. *Introduction*

Large scale invariant sets of planar diffeomorphisms can vary discontinuously in size with changes in parameters. Global bifurcations of observable sets, such as crises of attractors or metamorphoses of basin boundaries, are the most easily detected and probably the most often described in scientific literature. (For a partial list of references, see [**12**].) For example, Figure 1 shows a jump in the size of a chaotic attractor of the Ikeda map, as the attractor merges with an unstable invariant Cantor set. At the bifurcation parameter, it is not just that two invariant sets merge to form a larger attractor. Gaps in the unstable invariant set fill in suddenly as the bifurcation parameter is passed. These gaps do not fill in gradually: they are of positive width larger than a uniform constant for every parameter prior to bifurcation. For a further discussion of this example, see Robert *et al* [**11**].

The key to the investigation of changes in invariant sets is the set of *recurrent points*, i.e. points x such that $x \in \omega(x)$. The discontinuous appearance of new recurrent points at global bifurcations, as occurs in the gaps in the example above, is called an *explosion* in the recurrent set. Explosions in the non-wandering set, called Ω-*explosions*, are described in [**8**]. In our context, it is more natural to work with the set of chain recurrent points, which includes both the set of non-wandering points and the set of recurrent points.

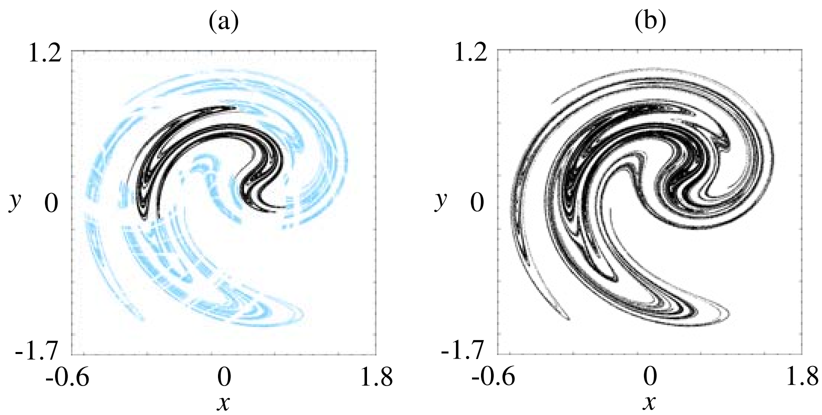

FIGURE 1. Explosions in the Ikeda map. Invariant sets for the Ikeda map $f_\lambda : \mathbb{R}^2 \to \mathbb{R}^2$ before and after bifurcation. The map is given by $f_\lambda(x, y) = (0.85 + 0.9(x\cos\tau_\lambda - y\sin\tau_\lambda), 0.85 + 0.9(x\sin\tau_\lambda + y\cos\tau_\lambda))$, where $\tau_\lambda = 0.4 - \lambda/(1 + x^2 + y^2)$. The black set shown in (a) is a numerically calculated orbit filling out an observed chaotic attractor at $\lambda = \lambda_0 \approx 7.268\,848\,94$. For every $\lambda > \lambda_0$, the orbit occupies the larger attractor shown in black in (b). This is a discontinuity in the size of the chaotic attractor. In addition to the attractor, (a) also contains an unstable invariant Cantor set, as shown in gray. Note that there are gaps in the saddle. In (b) all points of the saddle are contained in the attractor. The gaps fill in discontinuously at λ_0; the points in the gaps are explosion points at $\lambda = \lambda_0$.

We concentrate on explosions in the chain recurrent set, which we refer to as *chain explosions*. (The definition of a chain explosion appears in §2.) In fact, all of the chain explosions treated in this paper are also explosions in both the non-wandering and recurrent sets. This is discussed at greater length in the closing remarks.

It has often been noted that explosions may occur when there is a tangency between stable and unstable manifolds. When is a point of homoclinic tangency an explosion point? Palis and Takens [**8**] gave a partial answer to this question in their classification of homoclinic Ω-explosions. Let p_λ be a saddle fixed point for a dissipative family of orientation preserving planar maps f_λ with a homoclinic tangency. Assume coordinates have been chosen so that p_λ is at the origin, the stable manifold is locally the horizontal axis and the tangency point lies on the upper branch of the unstable manifold. Palis and Takens showed that if the family is area contracting, then generically if the upper branch of the unstable manifold approaches tangency from the upper half plane as parameter λ varies, then the tangency point is not an explosion point (see Figure 2). More generally, Palis and Takens gave conditions on the placement of the tangency point, sign of eigenvalues and area contraction or expansion of the map, under which an explosion does *not* occur at the point of homoclinic tangency. They assumed that prior to the bifurcation parameter, the map is persistently hyperbolic. They did not give sufficient conditions for an explosion, though in each of the cases which they could not rule out, they showed that there exist examples such that the introduction of a point of homoclinic tangency is an explosion at the bifurcation parameter.

In this paper, for a certain class of maps we give both necessary and sufficient conditions for heteroclinic tangencies to be explosion points in two dimensions. A heteroclinic tangency point is an explosion point only under rather restrictive conditions; under our

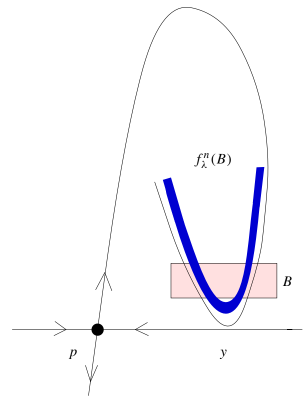

FIGURE 2. A non-explosion. This homoclinic tangency at y for an area–contracting map f_λ, has no sudden jump in the recurrent set. There is an invariant Cantor set of points in $B \cap f_\lambda^n(B)$ that are recurrent prior to tangency. The tangency point is a limit of these recurrent points.

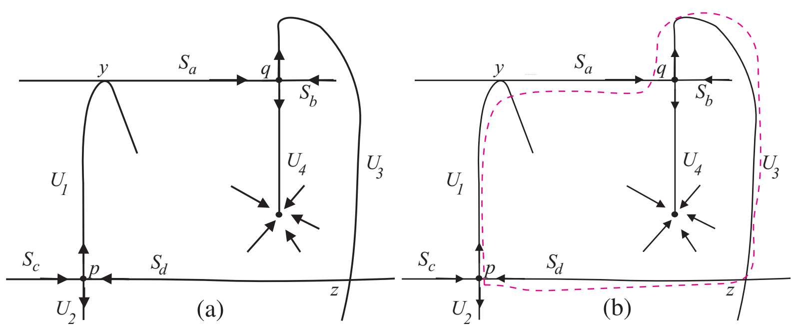

FIGURE 3. An explosion in the heteroclinic case. (a) An explosion occurs at y as a parameter is varied. The unstable branch U_4 is shown in the basin of an attracting fixed point. Therefore it does not cross the stable manifold of p. The point y is contained in a unique 'cycle' (Definition 3.4). That is, y is contained in the sequence of points (p, y, q, z, p), which alternate between fixed and heteroclinic. The first and last points are the same, and consecutive points are connected by compact connected pieces of stable and unstable manifolds. The path of the cycle is the union of the connecting pieces of stable and unstable manifolds. (b) The occurrence of an explosion corresponds to the fact that this is a 'crossing cycle'. That is, any perturbation of the path of the cycle, such as the dotted line, always crosses both $W^u(p)$ and $W^s(q)$.

hypotheses, the configuration shown in Figure 3(a) gives rise to an explosion as a parameter is varied. In contrast, the configuration shown in Figure 4(a) does not give rise to an explosion. In fact, in Figure 4 there are transverse homoclinic points arbitrarily near y. This implies the existence of recurrent points arbitrarily near y *before* tangency. This is not necessarily true of the situation in Figure 3. When there is a unique 'cycle' of alternating stable and unstable manifolds through y (Definition 3.4), it is possible to give a heuristic description of the difference between these two situations. Consider the closed curve formed by the manifold branches connecting y, q, z, and p. In the first case, *any* perturbation of this closed curve is another closed curve which crosses both $W^u(p)$ and $W^s(q)$, see the dotted line in Figure 3(b). This is a crossing cycle (Definition 3.7). The fact that there are no other cycles containing y (Definition 3.8, 'crossing point') implies that

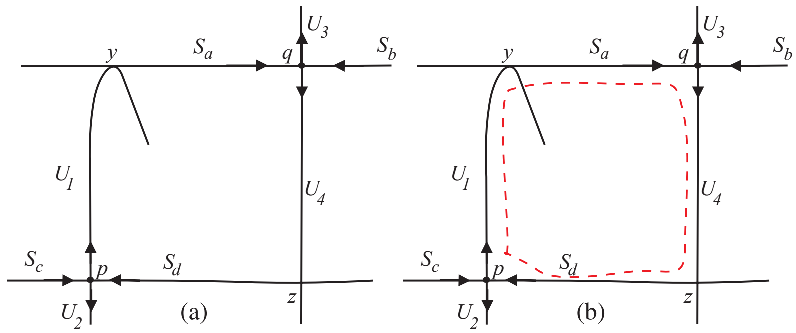

FIGURE 4. A non-explosion in the heteroclinic case. (a) No explosion occurs at y. (b) The lack of explosion corresponds to the fact that there are closed curves near the path of the cycle through y, such as the dotted line, which do not cross $W^s(q)$.

there is an explosion (Theorem 5.2). In contrast, in the second case it is possible to perturb the closed curve of manifold branches from y to itself so that the new curve does not cross *both* $W^u(p)$ and $W^s(q)$. The dotted line shown in Figure 4(b) is such a curve; it does not cross $W^s(q)$. This is not a crossing cycle and can never give rise to an explosion (Theorem 4.1). For more details, see §§3–5.

In investigating the heteroclinic case, in contrast to Palis and Takens' work on explosions at homoclinic tangencies, we make no assumption about area contraction, expansion, or on persistence of hyperbolicity. Rather, we consider which branches of manifolds are involved (see the definition of a crossing point) and the way in which these branches intersect (see H4 and H4′).

The rest of the paper proceeds as follows. Sections 2 and 3 contain definitions and hypotheses. Section 4 contains necessary conditions for heteroclinic points to be chain explosions. Section 5 contains sufficient conditions for chain explosions to occur at heteroclinic tangencies. The theorems in §4 and §5 are stated in terms of heteroclinic points between fixed point saddles. In §6 we examine more completely the structure of invariant manifolds within the chain classes of fixed points. In §7 the case of heteroclinic points between periodic points is discussed. In §8, we explain how our results apply to explosions in the non-wandering and recurrent sets.

2. *Changes in the chain recurrent set*

We now give a series of definitions related to the notion of chain recurrence.

Definition 2.1. For an iterated function g, there is an ε*-chain* from x to y when there is a finite sequence $\{z_0, z_1, \ldots, z_N\}$ such that $z_0 = x$, $z_N = y$, and $d(g(z_{n-1}), z_n) < \varepsilon$ for all n.

If there is an ε-chain from x to itself for every $\varepsilon > 0$ (where $N > 0$), then x is said to be *chain recurrent* [**4, 5**]. The *chain recurrent set* is the set of all chain recurrent points. For a one-parameter family f_λ, we say (x, λ) is chain recurrent if x is chain recurrent for f_λ.

If for every $\varepsilon > 0$, there is an ε-chain from x to y and an ε-chain from y to x, then x and y are said to be in the same *chain component* of the chain recurrent set.

Throughout the paper, we use the following notation and assumptions.

(H1) $f : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ is a C^1-smooth one-parameter family of C^2 diffeomorphisms.

In the homoclinic case, if f satisfies H1, and if the stable and unstable manifolds to a fixed point saddle p intersect only at p for $\lambda < \lambda_0$, intersect in a tangency at $\lambda = \lambda_0$, and cross for $\lambda > \lambda_0$, then there is a bifurcation in the chain recurrent set of f_λ at $\lambda = \lambda_0$. If no other intersections between stable and unstable manifolds occur, then there are no homoclinic orbits before λ_0, so the saddle fixed point is its own chain component. At λ_0, all points in the homoclinic orbit become part of this chain component. This sudden jump in the chain component of the fixed point is sometimes, but not always, a sudden jump in the entire chain recurrent set, since the newly formed homoclinic points may be limits of points in another chain component (see Figure 2, in which a homoclinic tangency produces a merging of two pre-existing chain components). Distinguishing between jumps in the chain recurrent set and jumps in chain components is quite important, as physically it corresponds to whether or not it is possible to predict bifurcations using the chain recurrent set.

In the following definition, we describe jumps in the entire chain recurrent set, as opposed to jumps in components (as shown in Figure 2; cf. 'Ω-explosion' in §8).

Definition 2.2. (Chain explosions) A *chain explosion point* (x, λ_0) is a point such that x is chain recurrent for f_{λ_0}, but there is a neighborhood N of x such that on one side of λ_0 (i.e. either for all $\lambda < \lambda_0$ or for all $\lambda > \lambda_0$), no point in N is chain recurrent for f_λ. Note that at f_{λ_0}, x is not necessarily an isolated point of the chain recurrent set.

3. *Bifurcations through tangency*

We now consider the case of a chain explosion at a heteroclinic point. Similar concepts have been previously explored by Hurley [**7**] and Patterson [**10**]. We assume the existence of a non-degenerate heteroclinic tangency at the bifurcation parameter. This is made precise by the following assumptions.

(H2) f_{λ_0} has (at least) two hyperbolic saddle fixed points called p and q. There is a unique continuation of these points for all nearby λ, so we will also refer to their continuations as p and q.

Remark 3.1. Section 7 describes the case in which p and q are periodic points rather than fixed points.

Definition 3.2. (Generic tangency point) Let (y, λ_0) be an intersection point between a branch of the unstable manifold $W^u(p)$ and a branch of the stable manifold $W^s(q)$. (A branch of manifold $W(r)$ is a connected component of $W(r) \setminus r$.) (y, λ_0) is a *generic tangency point* if for f_{λ_0}, there is a tangency at y between $W^u(p)$ and $W^s(q)$, locally, the manifolds do not cross and the two manifolds are not identical on some neighborhood. On one side of λ_0, the local pieces of manifold intersect transversally, and on the other side of λ_0, the manifolds do not locally intersect. Furthermore, at parameter λ_0, the orbit of tangency of y is unique: there are no other orbits of tangencies between stable and unstable manifolds of fixed points.

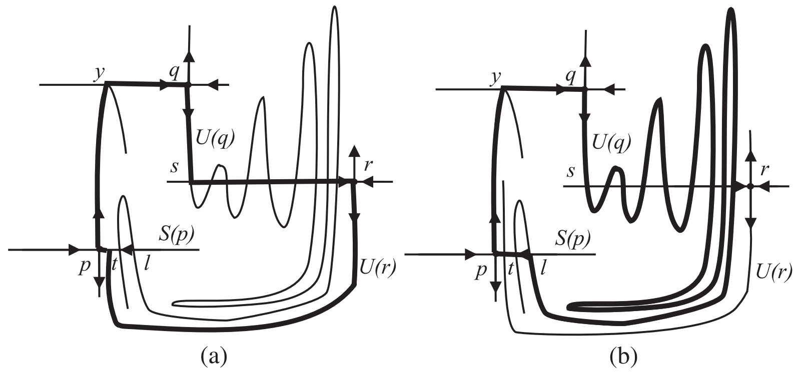

FIGURE 5. Four-point cycles. We only consider four-point cycles: two fixed points, one transverse heteroclinic point and one heteroclinic tangency point. Any cycle with a tangency that has more than four points, such as the six in (a), results in a four-point cycle (b). In this figure, (a) has thick lines to show the branches connecting cycle $\{p, y, q, s, r, t, p\}$; branch $U(q)$ limits on branch $U(r)$ and thus crosses branch $S(p)$. (b) The thick lines connecting the corresponding four-point cycle $\{p, y, q, l, p\}$ are shown.

(H3) Assume that (y, λ_0) is a generic tangency point.

In order for the tangency point (y, λ_0) to be chain recurrent, ε-chains must return to y. One natural way for this to occur is for there to be an intersection between a branch of $W^u(q)$ and a branch of $W^s(p)$. Specifically, we give a definition of an n-connection, as a precursor to the introduction of cycles [**9**].

Definition 3.3. (n-connection) An *n-connection* is a sequence of points $\{\varrho_1, t_1, \varrho_2, t_2, \dots, \varrho_n\}$ such that for all i, ϱ_i is a fixed point, and for each $i < n$, t_i is a heteroclinic point such that $t_i \in W^u(\varrho_i) \cap W^s(\varrho_{i+1})$.

Definition 3.4. (Cycle) A *cycle* is an n-connection $\{\varrho_1, t_1, \varrho_2, t_2, \dots, \varrho_n\}$ such that $\varrho_n = \varrho_1$.

Remark 3.5. In what follows, we consider cycles consisting of only two fixed points and two heteroclinic points. Under assumption H3, the fact that we only consider this type of cycles is not a restriction, since as a consequence of the λ-lemma, if $W^u(q)$ and $W^s(r)$ cross, and $W^u(r)$ and $W^s(p)$ cross, then $W^u(q)$ and $W^s(p)$ cross [**6, 12**]. Therefore, the existence of any cycle containing a tangency implies the existence of a cycle with four distinct points containing that same tangency. Technically, although this cycle is a three-connection, we refer to it as a 'four-point cycle' (see Figure 5).

A cycle consists only of heteroclinic and fixed points which are corner points on an alternating sequence of branches of stable and unstable manifolds. In the course of this paper, we also keep track of which manifold branches contain each point of a cycle, as described in the following remark.

Remark 3.6. (Labeling and placement of branches) If $\{p, y, q, z, p\}$ forms a cycle at $\lambda = \lambda_0$, then by hypothesis H3, the orbit of y is the only orbit of tangency, so the manifolds must cross at z. It is possible to distinguish branches of stable and unstable manifolds according

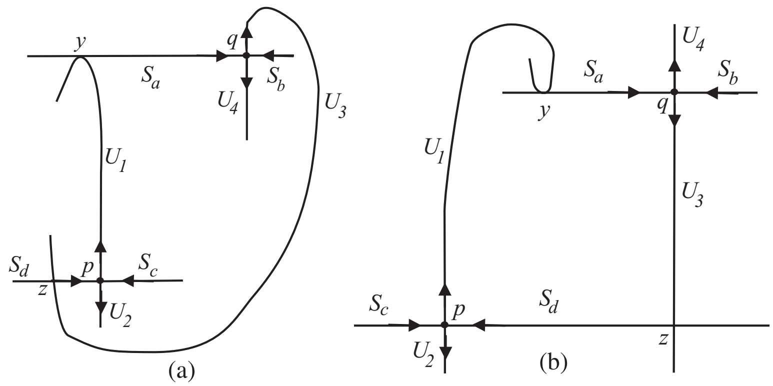

FIGURE 6. Crossing cycles. Cycles $\{p, y, q, z, p\}$ contain a tangency. In each cycle, iterates of U_1 in a neighborhood of y converge to U_4. That is, U_1 and U_4 are on the same side of S_a. Likewise, backward iterates of S_a in a neighborhood of y converge to S_c. Thus S_a and S_c are on the same side of U_1. In both (a) and (b), cycles $\{p, y, q, z, p\}$ are crossing cycles. Points y and z are candidates for crossing points.

to their placement with respect to the tangency point y. In this remark, we describe the labeling of branches when p and q have positive eigenvalues. We discuss the negative eigenvalue case in Remark 3.9.

Label the branch of $W^u(p)$ containing the tangency point y as U_1. Label the other branch of $W^u(p)$ as U_2. Let the branch of $W^s(q)$ containing y be S_a, and let the other branch be S_b.

We still need to distinguish branches of $W^u(q)$ and of $W^s(p)$. Since U_1 does not cross S_a at y by hypothesis H3, one can describe the branch of $W^u(q)$ on the *same side* of S_a as U_1 by looking at the location of forward iterates of local pieces of U_1. Namely, if iterates of local pieces of U_1 near y converge to a branch of $W^u(q)$, then U_1 and that branch are on the same side of S_a. If two branches are not on the same side of S_a, they are on the *opposite side* of S_a. That is, one branch of $W^u(q)$ is on the same side of S_a as U_1 and the other on the opposite side. Give the label U_4 to the branch of $W^u(q)$ on the same side of the segment of S_a as U_1. Give the label U_3 to the opposite side branch.

Similarly we can distinguish the two branches of $W^s(p)$ as being on the same side or opposite side of U_1 as S_a by looking at the location of backward iterates of local pieces of S_a near y. Let S_c be the branch of $W^s(p)$ on the same side of the segment of U_1 as S_a. Let the opposite side branch be S_d (see Figures 6 and 7).

Define $S(z)$ to be the branch of $W^s(p)$ containing z (either S_c or S_d). Define $U(z)$ to be the branch of $W^u(q)$ containing z (either U_3 or U_4).

Definition 3.7. (Crossing cycle) Consider a cycle $\{p, y, q, z, p\}$ containing p and q, where f satisfies H1, p and q satisfy H2 and have positive eigenvalues, and y is a tangency point satisfying H3. (We discuss the negative eigenvalue case in Remark 3.9.) This cycle is a *crossing cycle* if the following two conditions are satisfied. (a) The branch of $W^u(q)$ containing z and the branch of $W^u(p)$ containing y are on opposite sides of the branch

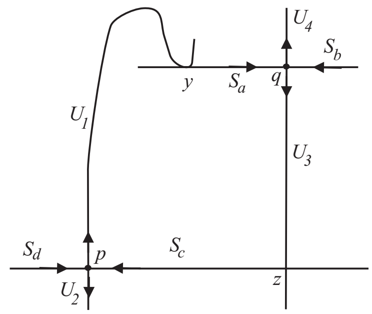

FIGURE 7. Non-crossing cycles. This cycle is not a crossing cycle. Hence by Theorem 4.1, neither y nor z is an explosion point.

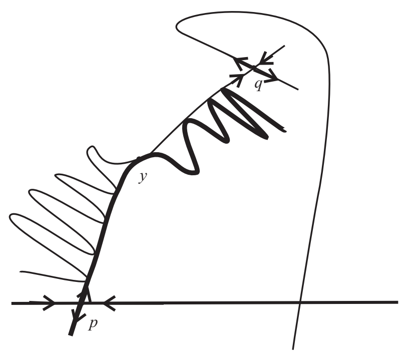

FIGURE 8. Symmetry of the conditions for a crossing cycle. If f has a crossing cycle, then so does f^{-1}.

of $W^s(q)$ containing y. (b) The branch of $W^s(p)$ containing z and the branch of $W^s(q)$ containing y are on opposite sides of the branch of $W^u(p)$ containing y.

Using the branch labels listed above in Remark 3.6, a cycle $\{p, y, q, z, p\}$ is a crossing cycle if (a) $U(z) = U_3$, and (b) $S(z) = S_d$. That is, y is a tangency in $U_1 \cap S_a$, and $z \in U_3 \cap S_d$.

The two conditions comprising the definition of a crossing cycle have a great deal of symmetry. Namely, if f has a crossing cycle, then f^{-1} also does. Replacing f by f^{-1} interchanges the labels U_1 and S_a as well as the labels U_3 and S_d (see Figure 8).

Definition 3.8. (Crossing point) A heteroclinic point is called a *crossing point* of f_{λ_0} if it is contained exclusively in crossing cycles at $\lambda = \lambda_0$. That is, it is contained in a crossing cycle and each cycle containing it is a crossing cycle (see Figure 6).

Remark 3.9. If p and/or q have negative eigenvalues, it is still possible to define a crossing cycle. The branches S_a, S_b, U_1 and U_2 are defined as before. The other branches are more complicated. For instance, both branches of $W^u(q)$ may be on the same side of S_a as U_1 (i.e. iterates of U_1 near y converge to both branches of $W^u(q)$). This is not a serious problem; in the negative eigenvalue case, a cycle is a crossing cycle when (a) $U(z)$ and U_1 are not on the same side of S_a and (b) $S(z)$ and S_a are not on the same side of U_1.

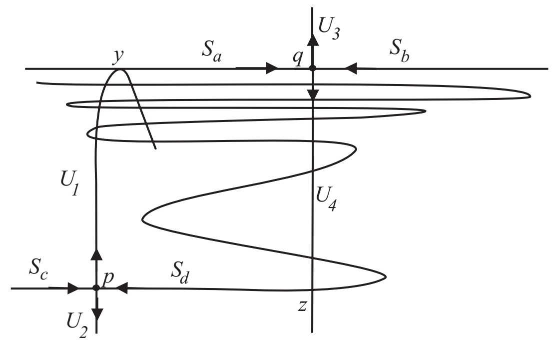

FIGURE 9. Horseshoes in non-crossing cycles. In a non-crossing cycle, there are transverse homoclinic points, and thus hyperbolic periodic points, converging to the tangency point. Therefore the tangency point cannot be a chain explosion point.

In some cases of negative eigenvalues, there may never be a crossing cycle. Specifically, in a cycle where q has an eigenvalue less than -1, the branch or branches of $W^u(p)$ containing the orbit of y limit on both branches of $W^u(q)$. Therefore U_1 is on the same side as both branches of $W^u(q)$, and the cycle is never a crossing cycle. Likewise, if p has an eigenvalue between 0 and -1, then the branch or branches of $W^s(q)$ containing the orbit of y limit on both branches of $W^s(p)$. That is, the branch of $W^s(q)$ containing y is on the same side as both branches of $W^s(p)$. The cycle is never a crossing cycle. Due to the fact that f is a diffeomorphism, we know that the determinant of the Jacobian always has the same sign. Therefore, there are seven cases in which p and/or q has a least one negative eigenvalue. Of these, the only case for which there may be a crossing point is that in which p has an eigenvalue less than -1, q has an eigenvalue between 0 and -1, and the other two eigenvalues are positive.

4. *Necessary conditions for chain explosions*

The relationship between a crossing cycle and a chain explosion results from the fact that whenever a cycle is not a crossing cycle, stable and unstable manifolds of the same point can access each other, and end up intersecting transversally infinitely often near the tangency point. Therefore, there are periodic points converging to the tangency point, implying that the tangency point is not a chain explosion point (see Figure 9). This is the content of the following theorem, which gives necessary conditions for a chain explosion at a heteroclinic point. The theorem is applied by observing that if a heteroclinic point is not a crossing point, then it is not an explosion point.

THEOREM 4.1. *Assume that f satisfies H1–H3, that (x, λ_0) is a chain explosion point, and that x is a heteroclinic point contained in a cycle of f_{λ_0}. Then x is in a cycle with a tangency, and x is a crossing point of f_{λ_0}.*

Remark 4.2. This theorem also holds for periodic points with the definitions given in §7.

Remark 4.3. Note that the point x satisfying the hypothesis of the above theorem may be contained in multiple cycles. That is, x may be contained in two cycles such that not all

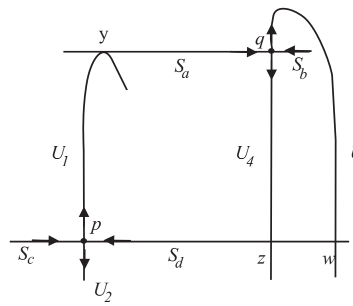

FIGURE 10. Coexistence of crossing and non-crossing cycles. A crossing cycle $\{p, y, q, w, p\}$ and a non-crossing cycle $\{p, y, q, z, p\}$ through the same tangency point y. The tangency point y is not a crossing point. The transverse heteroclinic point w is a crossing point.

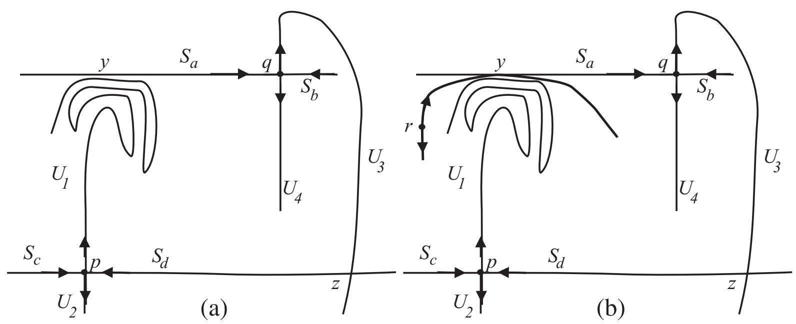

FIGURE 11. Limit points give rise to tangencies. (a) The case of point y being a limit of points on U_1. (b) We believe that in general, unless U_1 and S_a cross near y, y is a tangency point between S_a and an unstable manifold for some other periodic point r. This case is discussed in §6.

of the points of the second cycle are iterates of points of the first cycle. For example, the point y in Figure 10 is in both a crossing cycle and a non-crossing cycle. A chain explosion point may be a non-tangency point, even though the chain explosion comes as a result of a tangency. In Figure 10, points y and z are not crossing points, but w is a crossing point. The situation shown in Figure 10 illustrates the mechanism causing the global bifurcation shown in Figure 1 [**11**]. At the bifurcation parameter for the Ikeda attractor, there are chain explosions as a result of a tangency, though the tangency point itself is not a chain explosion point.

Remark 4.4. By looking at cycles, we are restricting to the case of intersection between unstable manifold $W^u(p)$ and stable manifold $W^s(q)$. Another common case is one in which the unstable manifold $W^u(p)$ limits on the stable manifold $W^s(q)$ at point y (see Figure 11). We believe that if there are no prior intersections between the stable and unstable manifolds, then in general the point y is in fact a point of tangency between stable manifold $W^s(q)$ and some other unstable manifold $W^u(r)$, for a fixed or periodic point r. See the discussion in §6.

Proof of Theorem 4.1. We give the proof for the case in which all eigenvalues are positive, commenting on other cases at the end of the proof.

Assume that x is contained in a cycle of f_{λ_0}. Label points as described in Remark 3.6. The reader may wish to refer to Figures 6 and 7 for this labeling scheme.

First, suppose the cycle containing x does not have a tangency. If heteroclinic points in a cycle are all at points at which stable and unstable manifolds cross, then there exist hyperbolic periodic points arbitrarily near every heteroclinic point in the cycle at parameter λ_0. These points persist under any small perturbation of f_{λ_0}. Thus (x, λ_0) is not a chain explosion point.

Now suppose that x is on a cycle with a tangency, and x is not a crossing point of f_{λ_0}. Then x is contained in a non-crossing cycle. There are two types of non-crossing cycles with tangencies: (1) a cycle containing a point at which U_4 and $W^s(p)$ intersect transversally; (2) a cycle containing a point at which S_c and $W^u(q)$ intersect transversally. The non-crossing point x is either the tangency point (which we label y) or the point at which $W^u(q)$ and $W^s(p)$ cross (which we label z) of one of these two types of cycles. In the case of type (1), the λ-lemma implies that pieces of $W^s(p)$ converge to $W^s(q)$ on the same side as U_4. However, by definition of U_4, this is on the same side of S_a as U_1. Therefore, pieces of $W^s(p)$ cross U_1 arbitrarily near tangency point y. Thus there are hyperbolic periodic points near y. These will persist under any small perturbation of f_{λ_0}. Therefore (y, λ_0) is not a chain explosion point.

In addition, by a modification of the λ-lemma, pieces of U_1 intersect S_d arbitrarily close to z. This implies that (z, λ_0) is not a chain explosion point.

Similarly, in the case of a type (2) cycle, the λ-lemma implies that pieces of $W^u(q)$ converge to $W^u(p)$ on the same side as S_c. However, by the definition of S_c, this is on the same side of U_1 as branch S_a. Therefore $W^u(q)$ crosses S_a arbitrarily near y, and (y, λ_0) is not a chain explosion point. Furthermore, the λ-lemma also implies that $W^u(q)$ converges to itself, and therefore crosses $W^s(p)$ arbitrarily close to z, implying that (z, λ_0) is not a chain explosion point. Therefore every cycle containing x must be a crossing cycle. This completes the proof, assuming all eigenvalues are positive.

In the case in which at least one eigenvalue of p or q is negative, let x be on a non-crossing cycle of f at λ_0. Then either x or $f(x)$ is on a non-crossing cycle for f^2. By the argument above for positive eigenvalues, either (x, λ_0) or $(f(x), \lambda_0)$ is not a chain explosion point for f^2. In either case, (x, λ_0) is not a chain explosion point for f. □

5. *Sufficient conditions for chain explosions at tangency*

We need a precise way of describing the existence of ε-chains from one point to another. This motivates the following definition.

Definition 5.1. Define the *positive chain set* by

$$\mathrm{Ch}^+(u) = \{x : \text{ there exists an } \varepsilon\text{-chain from } u \text{ to } x \text{ for all } \varepsilon > 0\},$$

and the *negative chain set* by

$$\mathrm{Ch}^-(v) = \{x : \text{ there exists an } \varepsilon\text{-chain from } x \text{ to } v \text{ for all } \varepsilon > 0\}.$$

Define the *chain set* from u to v as $\mathrm{Ch}(u, v) = \mathrm{Ch}^+(u) \cap \mathrm{Ch}^-(v)$.

We now state an assumption for the case in which $\mathrm{Ch}(a, b)$ is non-empty. The assumption is quite restrictive. However, in the next section we show that in our specific case of $a = q$ and $b = p$, the assumption is a consequence of a rather natural set of conditions on how points move from saddle to saddle under f when there is only one tangency. Note that this assumption specifies which branches of stable and unstable manifolds have to cross.

(H4 from a to b) Let a and b be hyperbolic saddle fixed points. If there are points of $\mathrm{Ch}(a, b)$ on branch $U(a)$, and if there are also points of $\mathrm{Ch}(a, b)$ on branch $S(b)$, then $U(a)$ crosses $S(b)$.

We need one more hypothesis before stating the main theorem of this section.

(H5: uniform bound on the chain recurrent set) There is a neighborhood Λ of λ_0 and a bounded set $V \subset \mathbb{R}^2$ such that for all $\lambda \in \Lambda$, the chain recurrent set of f_λ is contained in V.

THEOREM 5.2. *Assume f_λ satisfies H1–H3, H4 from q to p, and H5. Then the tangency point (y, λ_0) is a chain explosion point if and only if y is a crossing point for f_{λ_0}.*

This theorem is stronger than Theorem 4.1; for a point of tangency, it gives both necessary and sufficient conditions for a chain explosion to occur. In addition, we do not need to assume here that y is contained in a cycle. This follows from the hypotheses. Again, this theorem holds for periodic points. The appropriate definitions are given in §7.

The proof of Theorem 5.2 crucially relies on the following lemmas. The first lemma shows that ε-chains from a point u to a point v imply the existence of ε-chains from images and preimages of u to images and preimages of v.

LEMMA 5.3. *Assume g is a diffeomorphism. Let u and v be such that $v \in \mathrm{Ch}^+(u)$ and as $\varepsilon \to 0$, there are ε-chains from u to v which are arbitrarily long. Then for all integers k, $g^k(v) \in \mathrm{Ch}^+(u)$. Furthermore, all points in the forward and backward limit sets of v are contained in $\mathrm{Ch}^+(u)$.*

Similarly, if $u \in \mathrm{Ch}^-(v)$, then for all integers k, $g^k(u) \in \mathrm{Ch}^-(v)$. Furthermore, all points in the forward and backward limit sets of u are contained in $\mathrm{Ch}^-(v)$.

It is straightforward to show that as $\varepsilon \to 0$, there exist ε-chains which are arbitrarily long in the following two cases:

(a) *if u and v are not in the same orbit; or*

(b) *if $u = v$ is a chain recurrent point.*

Proof. If $v \in \mathrm{Ch}^+(u)$, then if $k > 0$, it is clear that $g^k(v) \in \mathrm{Ch}^+(u)$. Given ε, there is a δ small enough that if $\{u, z_1, \ldots, z_{N-2}, z_{N-1}, v\}$ is a δ-chain from u to v, then $\{u, z_1, \ldots, z_{N-2}, g^{-1}(v)\}$ is an ε-chain from u to $g^{-1}(v)$. This shows that $g^{-1}(v) \in \mathrm{Ch}^+(u)$. For any $k < 0$, the proof that $g^k(v) \in \mathrm{Ch}^+(u)$ follows recursively from the fact that $g^{-1}(v)$ is contained in the set. The inclusion of the forward and backward limit sets of v follows from the fact that $\mathrm{Ch}^+(u)$ is closed. The proofs of the rest of the statements are similar. □

The following lemma shows that a chain between fixed points implies a chain between points on unstable and stable manifolds of the fixed points.

LEMMA 5.4. *Assume that g is a C^2 diffeomorphism of the plane with hyperbolic saddle point a. If $z \in \mathrm{Ch}^+(a)$, $z \neq a$, then there is a point $u \in W^u(a)$, $u \neq a$, such that $u \in \mathrm{Ch}(a, z)$. Likewise, if $w \in \mathrm{Ch}^-(a)$, $w \neq a$, then there is a point $s \in W^s(a)$, $s \neq a$, such that $s \in \mathrm{Ch}(w, a)$.*

Proof. Choose a small neighborhood U of a not containing z such that g is conjugate to a linear saddle on U. It is possible to choose V, a compact subset of U not containing a (a ring around a), such that for sufficiently small ε, any ε-chain starting at a and leaving U must pass through V. Furthermore, as $\varepsilon \to 0$, ε-chains from a to z limit on $W^u(a)$ in V. This implies the existence of a point $u \in W^u(a)$ such that $u \in \mathrm{Ch}(a, z)$. The second case has a similar proof. □

The chain recurrent set is not necessarily closed under limits of functions, as the following example shows.

Example 5.5. Let $f_n : \mathbb{R}^2 \to \mathbb{R}^2$ be the identity outside the disk of radius n centered at the origin, and let each point inside the open disk be mapped to the right (with the vertical direction fixed). This can be done such that $\lim_{n\to\infty} f_n = f$ exists, and every point is mapped to the right by f, by a distance bounded away from zero. The chain recurrent set of f is empty. For all n, the chain recurrent set of f_n is the entire plane.

The next lemma states that under H5, limits of chain recurrent points are chain recurrent, even when the parameter is varied.

LEMMA 5.6. *Let g_λ be a continuous family of diffeomorphisms satisfying H5. If $\{(t_k, \lambda_k)\}$ is a sequence converging to (t_0, λ_0), and each t_k is chain recurrent for g_{λ_k}, then t_0 is chain recurrent for g_{λ_0}.*

Proof. Using the uniform bound on the chain recurrent set from H5, we assume that B is a compact set containing a one-ball of all chain recurrent points for all g_λ. For each k, let C_k be a periodic $1/k$-chain through t_k for the map g_{λ_k}. We can choose all C_k in B by the assumptions on B. C_k is a compact set. In the Hausdorff metric on compact sets, there is a subsequence of $\{C_k\}$ converging to a set $C \subset B$. C is closed, so $t_0 \in C$. Next we show that for every $q \in C$, and all $\varepsilon > 0$, there is an ε-chain for g_{λ_0} from q to itself, and this chain is contained entirely in C. At that point, we are done since $t_0 \in C$, implying t_0 is chain recurrent.

It remains to show the existence of a periodic ε-chain of g_{λ_0} from q to itself. First make the following choices of k and δ.

(1) Choose K_0 sufficiently large that if $k > K_0$ then on B, $d(g_{\lambda_k}, g_{\lambda_0}) < \varepsilon/4$. This is possible because g is continuous with respect to the parameter, so we have uniform continuity on compact sets.

(2) Choose $\delta < \varepsilon/4$ so that on B, if $d(x, y) < \delta$ then $d(g_{\lambda_0}(x), g_{\lambda_0}(y)) < \varepsilon/4$. Again, this is due to the uniform continuity on compact sets.

(3) Choose $k > K_0$ so that:
 (a) $1/k < \varepsilon/4$; and
 (b) $d(C_k, C) < \delta$;
 where distance is measured in the Hausdorff metric.

Recall that $C_k = (p_1^k, \ldots, p_j^k)$ is the periodic $\varepsilon/4$-chain through t_k. Choose points $x_1, \ldots, x_j$ in C such that $d(x_i, p_i^k) < \delta$ for all $1 \le i \le j$, and $x_i = q$ for some i. Extend this sequence periodically so that $x_{i+j} = x_i$. Then x_i is a periodic ε-chain for g_{λ_0} since

$$\begin{aligned} d(g_{\lambda_0}(x_i), x_{i+1}) &< d(g_{\lambda_0}(x_i), g_{\lambda_0}(p_i^k)) + d(g_{\lambda_0}(p_i^k), g_{\lambda_k}(p_i^k)) \\ &\quad + d(g_{\lambda_k}(p_i^k), p_{i+1}^k) + d(p_{i+1}^k, x_{i+1}) \\ &< \varepsilon. \end{aligned}$$

□

Proof of Theorem 5.2. We give the proof for positive eigenvalues, commenting on cases of negative eigenvalues at the end of the proof. We first show that if y is a chain explosion point, then y is contained in a cycle. If (y, λ_0) is a chain explosion point, then for $\lambda = \lambda_0$, y is chain recurrent. Since p is in the negative limit set of y, and q is in the positive limit set of y, by Lemma 5.3 with $u = v = y$, $\mathrm{Ch}(q, p)$ is non-empty. It follows from Lemma 5.4 that there are points $\tau \in W^s(p)$ and $\varphi \in W^u(q)$ such that $\varphi, \tau \in \mathrm{Ch}(q, p)$. By H4 from q to p the manifolds cross. Therefore y is contained in a four-point cycle. By Theorem 4.1, y is a crossing point.

Now we assume that (y, λ_0) is not a chain explosion point and show that y is not a crossing point at λ_0. If y does not lie on a crossing cycle at all, we are done. We assume that y is contained in a crossing cycle, and proceed to show that y must also be contained in some additional cycle.

Since y is contained in a crossing cycle at λ_0, it is chain recurrent. Furthermore, by hypothesis H3, there is a continuation y_λ for $\lambda > \lambda_0$ such that each y_λ is chain recurrent, being part of a transverse cycle. Since y is not a chain explosion point, there must be a sequence $\{(x_k, \lambda_k)\}$, converging to (y, λ_0) from one side (say $\lambda_k < \lambda_0$) such that each x_k is chain recurrent for f_{λ_k}.

Let $U_1, U_2, U_3, U_4, S_a, S_b, S_c$, and S_d be the respective manifold branches of p and q, where the labeling is as in Remark 3.6. Then there are three possibilities. First, there are an infinite number of points of $\{x_k\}$ on the U_4 side of the continuation of S_a. Second, there are an infinite number of $\{x_k\}$ on the continuation of S_a. Third, there are an infinite number of points of $\{x_k\}$ on the U_3 side of the continuation of S_a.

Case 1. Assume there are an infinite number of x_k on the U_4 side of the continuation of S_a. By continuity, any iterate of x_k is chain recurrent for f_{λ_k}. Since q is a saddle point for each parameter value, iterates z_k of the x_k's converge to a point z on U_4. (The x_k's are mapped to z_k's by different iterates of f for different k's.) By Lemma 5.6, z is chain recurrent at λ_0. In addition, z is in $\mathrm{Ch}^-(y)$ at λ_0, since for any ε, there is a k such that z_k is within ε of z, x_k is within ε of y, and there is an ε-chain from z_k to y (by continuity, this must be true for large enough k). Thus there is an ε-chain from z to y. This means that there are points on U_4 in $\mathrm{Ch}^-(p)$ at λ_0, since $f^{-k}(y)$ converges to p. By Lemma 5.4, there are points on U_4 and points on $W^s(p)$ such that the chain set between them is non-empty at λ_0. That is, there are points on U_4 and on a branch of $W^s(p)$ contained in $\mathrm{Ch}(q, p)$. Therefore by H4, U_4 crosses $W^s(p)$. This implies that y is in a non-crossing cycle.

Case 2. Assume there are an infinite number of x_k on the continuation of S_a. We know that at λ_0, inverse images of S_a converge to S_c. Therefore, by the same reasoning as in Case 1, there are points on S_c in $\mathrm{Ch}^+(y)$ at λ_0. Therefore y is in a non-crossing cycle.

Case 3. If there are an infinite number of x_k on the U_3 side of the continuation of S_a, then by an argument similar to Case 1, there are points on S_c in $\mathrm{Ch}^+(y)$ at λ_0. By the same reasoning as before, $W^u(q)$ crosses S_c at λ_0. Therefore y is in a non-crossing cycle.

This completes the proof for positive eigenvalues. In the case of negative eigenvalues at either p or q, assume that y is contained in a crossing cycle at λ_0 and that (y, λ_0) is not a chain explosion point for f. Then (y, λ_0) is not a chain explosion point for f^2. By the argument above for positive eigenvalues, y is contained in a non-crossing cycle for f^2 at λ_0. Therefore y is contained in a non-crossing cycle for f at λ_0. □

6. *Chains from a to b*

In this section, we explain the restrictive assumption H4. We start with a lemma describing precisely how the stable and unstable manifolds of a and b interact if there is an n-connection from a to b. This depends on the minimum number of points needed in an n-connection from a to b. The following result shows that an n-connection with only one tangency implies the existence of a connection with at most two fixed points on each side of the tangency.

LEMMA 6.1. *Assume that f is a C^2 diffeomorphism with hyperbolic fixed points, and at most one tangency of the type described in H3. The existence of an n-connection from hyperbolic fixed point a to fixed point b implies the existence of a two-, three-, or four-connection from a to b.*

Proof. As in Remark 3.5, if $U(\varrho_{i-1})$ crosses $S(\varrho_i)$, and $U(\varrho_i)$ crosses $S(\varrho_{i+1})$, then by the λ-lemma $U(\varrho_{i-1})$ crosses $S(\varrho_{i+1})$. If there is no tangency, then by the λ-lemma there is a connection $\{a, t, b\}$. Suppose t_k is a unique point of tangency (of the type in H3) in the n-connection $\{a, t_1, \varrho_2, t_2, \ldots, \varrho_k, t_k, \ldots, b\}$ and that there is no two- or three-connection. Then, by repeated use of the λ-lemma, $U(a)$ crosses $S(\varrho_k)$ at some point r, and $U(\varrho_{k+1})$ crosses $S(b)$ at some point s. Thus there is a four-connection $\{a, r, \varrho_k, t_k, \varrho_{k+1}, s, b\}$. □

LEMMA 6.2. *Let a and b be fixed points, and assume there is an n-connection from a to b. Under the hypotheses of Lemma 6.1, there is a two-, three-, or four-connection from a to b. Under these same hypotheses, if there is a two-connection from a to b, then the stable manifold $S(b)$ of b and the unstable manifold of $U(a)$ of a intersect. If the smallest connection between a and b is a three-connection, then $U(a)$ limits on points of $S(b)$. If the smallest connection between a and b is a four-connection $\{a, t_1, \varrho_2, t_2, \varrho_3, t_3, b\}$, then $U(a)$ limits on points of $S(\varrho_3)$, and there is a well-defined branch of $U(\varrho_3)$ containing t_3. This branch is on the opposite side of $S(\varrho_3)$ from $U(a)$.*

See Figure 12 for examples of all three possible connections. The proof of this lemma follows from repeated application of the λ-lemma, along similar lines to the proof of Lemma 6.1.

Knowing that the chain set from a to b is non-empty is not sufficient to imply the existence of an n-connection from a to b. However, we conjecture that generically such a non-empty chain set from one fixed point to another implies hypothesis H4′ below. Note that the truth of the above conjecture would imply that the case in which y is a limit point of $W^u(p)$ on $W^s(q)$, where the manifolds do not cross nearby, reduces to the

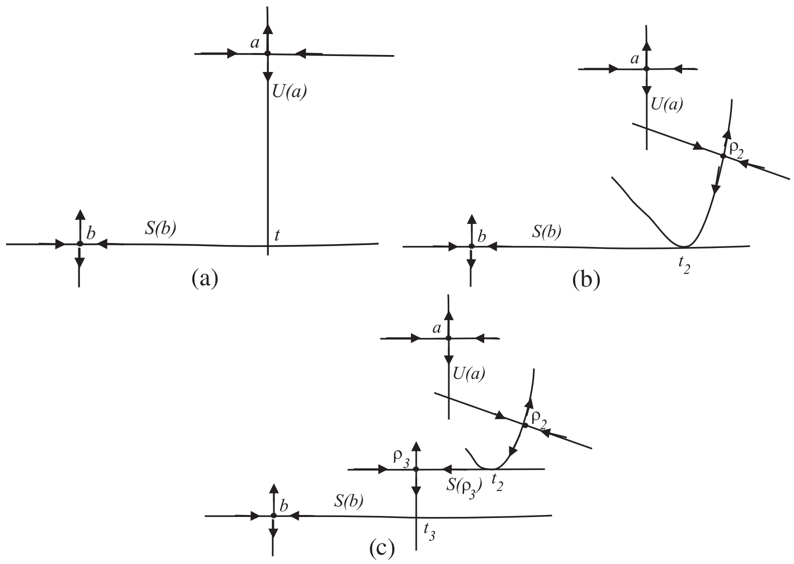

FIGURE 12. Situations in which the smallest n-connection from a to b is (a) a two-connection, (b) a three-connection, and (c) a four-connection. In (b) and (c), there is necessarily a tangency. In (c), $U(a)$ and $S(b)$ are on opposite sides of $S(\varrho_3)$.

case that y is a tangency between an unstable manifold $W^u(r)$ and stable manifold $W^s(q)$ (as in Figure 11). By Lemma 6.2, the following assumption is automatically satisfied if there is an n-connection from a to b.

(H4′: chain condition from a to b) Let a and b be hyperbolic saddle fixed points for a diffeomorphism. Assume that there are points in $\mathrm{Ch}(a, b)$ on branches $U(a)$ and $S(b)$. Then one of the following holds.

(a) Branch $U(a)$ intersects branch $S(b)$; they either cross or are tangent.

(b) Branch $U(a)$ limits on an orbit of branch $S(b)$, but does not intersect $S(b)$ along this orbit, and there are tangencies at points of this orbit. Points of the orbit are tangencies between $S(b)$ and another unstable manifold branch (see, for example, Figure 12(b)).

(c) There are two other saddle points ϱ_2 and ϱ_3 in $\mathrm{Ch}(a, b)$ such that branch $U(a)$ limits on a tangency t_2 between $U(\varrho_2)$ and $S(\varrho_3)$. Furthermore, there is a well-defined branch $U(\varrho_3)$ on the opposite side of $S(\varrho_3)$ from $U(a)$. This branch $U(\varrho_3)$ contains points of $\mathrm{Ch}(a, b)$, but no limit points of $U(a)$ (see, for example, Figure 12(c)).

These hypotheses include, but are not limited to, the cases shown in Figure 12; in (b) and (c) of the figure, $U(a)$ and $S(b)$ do not intersect, but the chain set $\mathrm{Ch}(a, b)$ contains a point of tangency. Note, however, that it is possible to satisfy the chain condition from a to b, where $\mathrm{Ch}(a, b)$ is non-empty, without having an n-connection from a to b.

LEMMA 6.3. *If f satisfies H1–H3, and H4′ from q to p, then f satisfies H4 from q to p.*

Proof. Assume that $\mathrm{Ch}(q, p)$ contains points on branches of $W^u(q)$ and $W^s(p)$. We label these branches $U_a(q)$ and $S_1(p)$ respectively. If H4′(a) holds, then the fact that there is a unique tangency means that $U_a(q)$ and $S_1(p)$ must cross, implying H4. If H4′(b) holds, then there is a tangency on $S_1(p)$. However, the unique tangency y is on $W^s(q)$. Stable manifolds do not intersect, thus H4′(b) does not occur. If H4′(c) holds, then $\varrho_2 = p$, and $\varrho_3 = q$. We know that $U_a(q)$ limits on $W^s(p)$, and the other branch of $W^u(q)$, which we denote $U_b(q)$, contains points of $\mathrm{Ch}(q, p)$. However, the chain condition holds again with $U(a) = U_b(q)$. This arrives at a contradiction, since there is no opposite side branch at q. Therefore condition (a) of H4′ must hold. □

Remark 6.4. (Functions satisfying the chain condition) If a one-parameter family f_λ is area-contracting, orientation-preserving, all fixed points are hyperbolic, the family satisfies H1–H3 and H5, and $W^u(q)$ is bounded, then we conjecture that Theorem 5.2 still holds if we replace hypothesis H4 (or H4′) by the following topological assumption, related to the notion of *prime end rotation number* [**3**]. The prime end rotation numbers associated with the branches of $W^u(q)$ are both zero, such as occurs with no previous crossings of the stable and unstable manifolds of q.

7. *Periodic points*

In this section, we consider heteroclinic tangencies between periodic points. Many of the results carry over without much comment to this case. Here are the necessary modifications, assuming the periodic orbits have positive eigenvalues.

(H2$^{\mathrm{p}}$) f_{λ_0} has (at least) two hyperbolic saddle periodic points p and q in distinct orbits of least periods n and m respectively.

Remark 7.1. The case in which p and q are in the same orbit is actually more closely related to the case of a homoclinic orbit than to that of a heteroclinic orbit, in that considerations of area contraction or expansion are significant. This is the ‘rotary tangency’ considered in [**1, 2**].

An *n-connection* for periodic orbits is precisely the same as the fixed point case, but with the fixed points replaced by periodic points. That is, it is a sequence $\{\varrho_1, t_1, \varrho_2, t_2, \ldots, \varrho_n\}$ so that t_i are as before, but ϱ_i are periodic. Likewise, a *cycle* is an n-connection with $\varrho_1 = \varrho_n$. As before, using the λ-lemma, we can assume that a cycle with a unique orbit of tangency contains periodic points from only two orbits.

The following is a definition of a crossing cycle for a cycle with one orbit of tangency and two distinct periodic points. It requires that at *every* point in the orbit of tangency, the local picture is like a crossing cycle for fixed points. That is, at every tangency point in the cycle, the branches of stable and unstable manifolds must be on opposite sides.

Definition 7.2. (Crossing cycle$^{\mathrm{p}}$) Consider a cycle with a unique orbit of tangency and two distinct periodic points with positive eigenvalues. This cycle is of the form $\{p_1, y_1, q_1, p_2, y_2, q_2, \ldots, y_N, q_N, p_N = p\}$. All p_i are in the same orbit, as are q_i and y_i. However, $f(p_r)$ is not necessarily equal to p_{r+1}. Let p be least period n and q least period m. This cycle is a *crossing cycle* if the following conditions hold for every

$k \leq N$. (1) The branch of $W^u(q_k)$ and the branch of $W^u(p_k)$ contained in the cycle are on opposite sides of $W^s(q_k)$. That is, under f^m, local pieces of the branch of $W^u(p_k)$ near y_k converge to q_k on the opposite side from the branch of $W^u(q_k)$ contained in the cycle. (2) The branch of $W^s(q_k)$ and the branch of $W^s(p_k)$ contained in the cycle are on opposite sides of $W^u(p_k)$. That is, under f^{-n}, local pieces of the branch of $W^s(q_k)$ near y_k converge to p_k on the opposite side from the branch of $W^s(p_k)$ contained in the cycle.

As before, a *crossing point* is a point exclusively contained in a crossing cycle. Using these new definitions, Theorem 4.1 holds in the periodic case.

THEOREM 7.3. *If a heteroclinic point is also an explosion point, then it is a crossing point.*

Proof. Assume that the point v is contained in a cycle that is not a crossing cycle. Then there is some iterate $f^r(v)$ such that the local picture is not that of a crossing cycle. This means either (1) the branch of $W^s(f^{-r}(p))$ in the cycle crosses $W^u(q)$ on the same side as the tangency branch of $W^u(p)$; or (2) the branch of $W^u(f^r(q))$ which crosses $W^s(p)$ is an image of the branch of $W^u(q)$ on the same side of $W^s(q)$ as the tangency branch of $W^u(p)$. In either case, this implies the existence of transverse homoclinic points converging to $f^r_{\lambda_0}(v)$. Thus $(f^r_{\lambda_0}(v), \lambda_0)$ is not a chain explosion point, which means that (v, λ_0) is not a chain explosion point. □

The restatement of Theorem 5.2 requires a version of the chain condition for periodic orbits as follows.

(H4$^{\mathrm{p}}$ from a to b) Let a and b be hyperbolic saddle periodic points. Assume that there are points of $\mathrm{Ch}(a, b)$ on branch $U(a)$. Assume also that there are points of $\mathrm{Ch}(a, b)$ on branch $S(b)$. Then an iterate of $U(a)$ crosses $S(b)$.

The conclusions of Theorem 5.2 remain valid, once the fixed point hypotheses are replaced by the corresponding periodic hypotheses.

THEOREM 7.4. *Assume that f_λ satisfies H1, H2^p, H3, H4^p from q to p, and H5. The tangency point (y, λ_0) is a chain explosion point if and only if it is a crossing point for f_{λ_0}.*

Proof. Assume the hypotheses above. Assume that at f_{λ_0}, the point y is a point of tangency between $W^u(p)$ and $W^s(q)$, but (y, λ_0) is not a chain explosion point. Using hypothesis H4$^{\mathrm{p}}$ and the same reasoning as in the fixed point case, either: (1) there is a branch U of some iterate $W^u(f^r(q))$, $1 \leq r \leq n$, of the branch of $W^u(q)$ on the same side of $W^s(q)$ as the tangency branch of $W^u(p)$, such that U crosses $W^s(p)$; or (2) the branch of $W^s(p)$ on the same side of $W^u(p)$ as $W^s(q)$ crosses $W^u(f^{-r}(q))$, where $1 \leq r \leq m$. It only remains to show that this non-crossing local picture fills out to a non-crossing cycle. This follows from the fact that iterates of branches of stable and unstable manifolds eventually map onto themselves again. □

8. *Closing remarks*

As mentioned in the introduction, previous work on explosions gives results for sudden changes in the non-wandering set [**8, 9**] at homoclinic tangency. For completeness, we give the definition of non-wandering points here. For more details, see Robinson [**12**].

Definition 8.1. (Non-wandering points) For a diffeomorphism g, a point x is *non-wandering* if for every neighborhood U of x, there is an $n > 0$ such that $g^n(U) \cap U$ is non-empty. The set of all non-wandering points for g is called the *non-wandering set*, denoted by $\Omega(g)$.

For all diffeomorphisms g, the recurrent set is contained in $\Omega(g)$, and $\Omega(g)$ is contained in the chain recurrent set. However, there may be chain recurrent points which are not contained in $\Omega(g)$. The proof of Theorem 4.1 shows that if there is a non-crossing cycle, then prior to tangency there are transverse homoclinic points converging to each heteroclinic point in the cycle. This means that the heteroclinic points are neither Ω-explosions nor explosions in the set of recurrent points. Under hypotheses H1–H5, if y is a crossing point for f_{λ_0}, then (y, λ_0) is a chain explosion point. Specifically, the proof shows that prior to tangency, there are no chain recurrent points near y, and thus no non-wandering or recurrent points. Furthermore, after passing through tangency, there are transverse heteroclinic points (contained in cycles) near y. Thus there are non-wandering and recurrent points near y. This implies y is an Ω-explosion point and an explosion point in the recurrent set. Therefore Theorems 4.1 and 5.2 hold for explosions in the non-wandering and recurrent sets.

We end with a comment on more general types of explosions. For planar diffeomorphisms, the situation of tangencies between stable and unstable manifolds to fixed and periodic points is by no means the most general situation. Chain explosions points can occur as a result of tangencies between generalized stable and unstable manifolds (to basic sets). They can also occur due to saddle-node type bifurcations, in which the saddle and node could each be part of a larger chain component. We conjecture, similar to the conjecture in [**9, 11**], that in the plane this list is complete. That is, all chain explosions occur as a result of either generalized saddle-node bifurcations or tangencies between stable and unstable manifolds.

Acknowledgements. We would like to thank Carl Robert for creating Figure 1 and Marcy Barge for his suggestions. Thanks also go to the referee, whose comments helped us make improvements to the paper, including Example 5.5. JY was partially supported by The National Science Foundation (Divisions of Math and Physics) and by The W. M. Keck Foundation.

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